Excess Returns, Average Returns, and the Adjustment Mechanism of the External Position of a Country - Supplementary Material

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Abstract

This document provides accessory material to the submitted version of the paper *Excess Returns, Average Returns, and the Adjustment Mechanism of the External Position of a Country.*

The main purpose of this document is to provide some accessory material to the published version of the paper *Excess Returns, Average Returns, and the Adjustment Mechanism of the External Position of a Country* as prepared for journal submission. I first provide a full description of the theoretical model that generates the simulated data used in the VAR section of the paper and some more details about the approximation of the external constraint. I then present some additional results for the empirical VAR analysis, especially an extended discussion of the analysis with simulated data. Finally, I provide some sensitivity analysis and robustness checks of the VAR estimation.

1 Theoretical Model

This Appendix provides a more exhaustive description of the theoretical model used to generate the simulated data for the VAR analysis in Section 4.3. The model is very similar in spirit to that used by Tille and van Wincoop (2010) too. I consider a model with two large countries, Home and Foreign ($H$and F), which are assumed to have similar, but not necessarily equivalent, sizes in steady state. The two countries have access to two international securities in order to hedge against aggregate country idiosyncratic shocks. One innovation is a productivity shock, the other is a shock to preferences affecting the intertemporal subjective

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discount factor. As a consequence, incomplete international financial markets are assumed. Given market incompleteness, an endogenous intertemporal discount factor that responds in a negative way to the aggregate consumption level is used in order to ensure the existence of a well defined steady state distribution of net wealth, and the other variables of the model.

The shocks to productivity introduce a differential in the rate of growth of productivity across regions; the shocks to preferences can be used to represent shifts in demand due, for instance, to aging of population. These are generally believed to be the causes of the currently observed international imbalances. Each country produces a single tradable good, which can be either consumed or exported to the other country.

I assume that the consumer has some degree of home bias in consumption. Agents set asset holdings for the current period based on the expected correlation of their returns with the endogenous consumption-based kernel. Production is exogenous and each country receives a stochastic endowment of the domestic good in each period. The two internationally traded assets represent claims on the output of each country. Prices are flexible and the law of one price holds. However, the home bias allows the real exchange rate to differ from parity. Under these assumptions, the dynamics of prices is fully summarized by the term of trade.

1.1 Output

I will focus on the home country H, the foreign country is assumed to have symmetric characteristics if not otherwise specified.

Production factors are assumed to be inelastically supplied by the representative household; they are constant and normalized to 1. Domestic good output, Y_t, follows a deterministic trend starting from an initial value Y_0, and it grows at a constant (log) gross growth rate g. This is an endowment economy and the amount of output available in each period is determined by a stochastic technological level, e^{ε_t}, in which $\varepsilon_t$follows an autoregressive process. Equation (1) illustrates how the level of output evolves

$$Y_t = Y_0 e^{gt+\varepsilon_t} \tag{1}$$

The technological level process is regulated by the autoregressive parameter ρ_ε, so $\varepsilon_t = \rho_\varepsilon \varepsilon_{t-1} + e_t$, where e_t is an *i.i.d.* productivity shock. The deterministic trend component of output is defined by $\bar{Y}_t = Y_0 e^{gt}$ and output can be simply re-written as $Y_t = \bar{Y}_t e^{\varepsilon_t}$. Finally, the foreign good output is similarly expressed as $Y_t^* = \bar{Y}_t^* e^{\varepsilon_t^*}$, where $\bar{Y}_t^* = Y_0^* e^{g^* t}$ and $\varepsilon_t^* = \rho_\varepsilon^* \varepsilon_{t-1}^* + e_t^*$ and a superscript star indicates foreign variables. I assume that the two countries are symmetric except for the initial values of output, which are allowed to differ if necessary. This is a condition required to have non-zero steady state net debt positions, as discussed below. Hence, we can assume that $g = g^*$ and $\rho_\varepsilon = \rho_\varepsilon^*$.

Let $P_{H,t}$ be the price of the H good in the H market and $P^*_{H,t}$ the corresponding price in the foreign F market expressed in the foreign currency; the equivalent prices for the F good are $P_{F,t}$ and $P^*_{F,t}$. Perfect exchange rate pass through is assumed; this implies that the law of one price (LOP) for the traded goods holds. Defining S_t, the nominal exchange rate between the two countries, as the price of currency F in terms of currency H, so that an increase of S is a depreciation of currency H, LOP implies $P_{H,t} = S_t P^*_{H,t}$ and $P_{F,t} = S_t P^*_{F,t}$. Good H can be consumed either at home or abroad; consumers can use the international financial assets to finance imports and enhance consumption if wanted and feasible.

The ratio $\frac{P_{H,t}}{P_t}$, where P_t is the domestic price index defined below, is endogenously determined in equilibrium at time t in order to guarantee that international markets clear and the general equilibrium of the model. Given the definition of P_t in (4), the ratio $\frac{P_{H,t}}{P_t}$ can be rewritten as

$$\frac{P_{H,t}}{P_t} = \frac{1}{\left[\lambda + (1-\lambda)\tau_t^{1-\theta}\right]^{\frac{1}{1-\theta}}} \tag{2}$$

where the term of trade τ_t is defined as

$$\tau_t = \frac{P_{F,t}}{P_{H,t}} = \frac{P^*_{F,t}}{P^*_{H,t}} \tag{3}$$

1.2 Consumption

The representative household consumes a composite good C_t defined by a CES function over the home and foreign good. All goods are tradable, but not perfect substitute, and the elasticity of substitution is given by the parameter ϕ

$$C_t = \left[\lambda^{\frac{1}{\phi}} (C_{H,t})^{\frac{\phi-1}{\phi}} + (1-\lambda)^{\frac{1}{\phi}} (C_{F,t})^{\frac{\phi-1}{\phi}}\right]^{\frac{\phi}{\phi-1}}$$

where $\lambda \in (\frac{1}{2}, 1)$ represents the home bias in consumption (assumed exogenously given). $C_{H,t}$ and $C_{F,t}$ are the H consumer's consumption of the good produced in the H country and of the good produced in the F country respectively. The corresponding for the F consumer is

$$C^*_t = \left[(1-\lambda)^{\frac{1}{\phi}} (C^*_{H,t})^{\frac{\phi-1}{\phi}} + \lambda^{\frac{1}{\phi}} (C^*_{F,t})^{\frac{\phi-1}{\phi}}\right]^{\frac{\phi}{\phi-1}}$$

The assumption about the home bias introduces an asymmetry in the definition of the consumption bundle that makes the CPIs of the two countries differ, even though the law of price holds. Those price indices P_t and P_t^* are

$$P_t = \left[\lambda P_{H,t}^{1-\phi} + (1-\lambda) P_{F,t}^{1-\phi}\right]^{\frac{1}{1-\phi}} \quad (4)$$

$$P_t^* = \left[(1-\lambda)(P_{H,t}^*)^{1-\phi} + \lambda (P_{F,t}^*)^{1-\phi}\right]^{\frac{1}{1-\phi}} \quad (5)$$

The PPP (purchasing power parity) does not hold and the real exchange rate γ_t does not necessarily have to be equal to 1

$$\gamma_t = \frac{S_t P_t^*}{P_t} \quad (6)$$

γ_t is defined as the price of the F good in terms of the H good; a decrease of γ_t corresponds to an appreciation of the H good/H RER (F good is becoming cheaper), while an increase of γ_t corresponds to a depreciation. From (6), it is easy to see how the real exchange rate γ_t is related to the term of trades τ_t by the equation

$$\gamma_t = \frac{\left[(1-\lambda) + \lambda \tau_t^{1-\theta}\right]^{\frac{1}{1-\theta}}}{\left[\lambda + (1-\lambda)\tau_t^{1-\theta}\right]^{\frac{1}{1-\theta}}} \quad (7)$$

The last elements coming from the intraperiod cost minimization problem of the consumer are the relative demand of H and F goods (here reported for the H consumer)

$$C_{H,t} = \lambda \left(\frac{P_{H,t}}{P_t}\right)^{-\phi} C_t \quad (8)$$

$$C_{F,t} = (1-\lambda) \left(\frac{P_{F,t}}{P_t}\right)^{-\phi} C_t \quad (9)$$

$C_{F,t}$ and $C_{H,t}^*$ are respectively the imports and exports of country H. For the F consumer the demand functions are symmetric: $C_{H,t}^* = (1-\lambda)\left(\frac{P_{H,t}^*}{P_t^*}\right)^{-\phi} C_t^*$ and $C_{F,t}^* = \lambda \left(\frac{P_{F,t}^*}{P_t^*}\right)^{-\phi} C_t^*$.

The representative household maximizes a time separable utility function defined over consumption. The lifetime utility is

$$\mathbb{E}_t \sum_{i=0}^{\infty} \theta_t^i U(C_{t+i})$$

where $\mathbb{E}_t$ is the expectation operator. The subjective intertemporal discount factor follows an Uzawa-type specification; θ_t endogenously responds to the level of consumption to ensure a stationary steady state distribution of wealth, since with incomplete markets the endogenous variables, in particular the wealth process, would be non stationary making any approximation method potentially imprecise. Formally θ_t is recursively defined starting from the uniperiodal time varying discount factor β_t as

$$\theta_t = \beta_t \theta_{t-1} \tag{10}$$

and β_t is assumed to be a decreasing function of the aggregate consumption in the previous period, taken as exogenously given by the consumer at the moment of his time t decision. I assume the following functional form for β_t

$$\log \frac{\beta_t}{\beta_{ss}} = \rho_\beta \log \frac{\beta_{t-1}}{\beta_{ss}} + \psi \log \frac{c_{t-1}}{c_{ss}} + u_t \tag{11}$$

where u_t is an *i.i.d.* demand shock and c_t is consumption normalized by output trend, while β_{ss} and c_{ss} are the steady state values of these two variables define below. The parameter ψ should be positive, but small in order to ensure the desired inverse relation between β and consumption, while still allowing for a smooth transition of wealth around the steady state. An equivalent form is assumed for β^*, which shares the same parameters and steady state value as β.

1.3 Asset Structure and Budget Constraint

Savings is allocated over two internationally traded securities that represent claims on domestic output, and can obviously generate capital gains and losses as their prices change over time. The payoffs depend on the realization of the shocks of the model. The total supply of each type of security is standardized to 1; each domestic agent owns the whole claim to domestic output and chooses the share of it to retain.

Let $V_{L,t}$ be the share of the H-security that consumer H chooses to hold from the beginning of period t to the beginning of $t+1$; $V_{A,t}$ is the H consumer's share of the F security. The analogous for the F consumer would be $V^*_{L,t}$ and $V^*_{A,t}$. Here, A and L refer to the notation for foreign assets and domestic liabilities (from the point of view of the H consumer) used in the paper.

The budget constraint of the consumer in real terms is

$$(Z_{L,t} + Y_t)V_{L,t-1} + \gamma_t(Z^*_{A,t} + Y^*_t)V_{A,t-1} = Z_{L,t}V_{L,t} + \gamma_t Z^*_{A,t}V_{A,t} + C_t \tag{12}$$

where $Z_{L,t}$ and $Z^*_{A,t}$ are the prices of one share of the H and F indices expressed in local currencies and, again, γ_t is the nominal exchange rate. Subtracting $(Z_{L,t} + Y_t)$ on both sides of the equation and after re-adjusting a couple of terms, the constraint is equivalently expressed as

$$-(Z_{L,t} + Y_t)(1 - V_{L,t-1}) + \gamma_t(Z^*_{A,t} + Y^*_t)V_{A,t-1} = -Z_{L,t}(1 - V_{L,t}) + \gamma_t Z^*_{A,t}V_{A,t} + C_t - Y_t$$

We can use now the definitions of assets $A_t = \gamma_t Z^*_{A,t} V_{A,t}$ and liabilities $L_t = Z_{L,t}(1 - V_{L,t})$ holdings of the H country and the expressions for the gross real asset returns $R_{L,t}$ and $R_{A,t}$

$$\begin{aligned} R_t^A &= \frac{\gamma_t}{\gamma_{t-1}} \left(\frac{Z^*_{A,t} + Y^*_t}{Z^*_{A,t-1}} \right) \\ R_t^L &= \frac{Z_{L,t} + Y_t}{Z_{L,t-1}} \end{aligned}$$

to finally rewrite the budget constraint in the same form as equation (1) in the main paper

$$L_t - A_t = R_t^L L_{t-1} - R_t^A A_{t-1} - (Y_t - C_t) \tag{13}$$

where, in this economy with no government spending and investment, we also have the trade surplus $D_t = Y_t - C_t$. A symmetric constraint is then implied by the marketing clearing conditions of assets and traded goods for the F country as well.

An alternative and convenient form to re-write constraint (13) in order to highlight excess and average returns is

$$B_t = R_t^M B_{t-1} - R_t^X B_{t-1}^M - D_t$$

which also allows us to apply the solution strategy of Devereux and Sutherland (2011) for this type of model, where the steady state values of asset holdings are not necessary to find the dynamics of net debt up to a first order accuracy.

1.4 Equilibrium Conditions, Market Clearing, and Normalization of the Model

I assume a CRRA utility function with relative risk aversion parameter σ. The first order conditions of the consumption maximization problem are easily derived for the H country

$$\mathbb{E}_t \left[\Lambda_{t,t+1} R_{t+1}^A \right] = 1 \tag{14}$$

$$\mathbb{E}_t \left[\Lambda_{t,t+1} R_{t+1}^L \right] = 1 \tag{15}$$

where $\Lambda_{t,t+1}$ is the consumption based stochastic discount factor defined as $\Lambda_{t,t+1} = \beta_t \left(\frac{C_t}{C_{t+1}} \right)^{-\sigma}$. Condition (14) and (15) can be combined to obtain the optimality condition in terms of the excess returns R_t^X

$$\mathbb{E}_t \left[\Lambda_{t,t+1} \left(R_{t+1}^A - R_{t+1}^L \right) \right] = \mathbb{E}_t \left[\Lambda_{t,t+1} R_{t+1}^X \right] = 0 \tag{16}$$

A first order linearization of this condition shows that the first order component of the excess returns is zero in expectation and R^X can be treated as an *i.i.d.* term in the solution (see Devereux and Sutherland, 2011).

A similar set of conditions hold for the F country, with the real exchange rate γ_t appearing in the portfolio conditions now and $\Lambda^*_{t,t+1} = \beta^*_t \left(\frac{C^*_t}{C^*_{t+1}}\right)^{-\sigma}$[1]

$$\mathbb{E}_t \left[\Lambda^*_{t,t+1} \left(\frac{\gamma_t}{\gamma_{t+1}}\right)^{-\sigma} R^A_{t+1}\right] = 1$$
$$\mathbb{E}_t \left[\Lambda^*_{t,t+1} \left(\frac{\gamma_t}{\gamma_{t+1}}\right)^{-\sigma} R^L_{t+1}\right] = 1$$

Market clearing conditions on the goods markets simply imply

$$Y_t = C_{H,t} + C^*_{H,t} = \left(\frac{P_{H,t}}{P_t}\right)^{-\sigma} \left[\lambda C_t + (1-\lambda)\gamma^\phi_t C^*_t\right]$$
$$Y^*_t = C_{F,t} + C^*_{F,t} = \left(\frac{P_{F,t}}{P_t}\right)^{-\sigma} \left[(1-\lambda) C_t + \lambda\gamma^\phi_t C^*_t\right]$$

where use have been done of the LOP condition, the definition in (6), and the optimal relative demand functions (8) and (9). On the assets market, with two countries, positions are opposite to each other; in particular, the external debt of the F country is defined as

$$B_t = -B^*_t$$

The deterministic trend of output makes the variables of the model growing over time. In order to solve the model with the standard linearization techniques, it is necessary to transform the variables in order to satisfy the balanced growth path requirement and have a well-defined steady state around which conduct the approximation of the model. The model is then normalized by dividing the variables by $\bar{Y}_t$; lower case notation will indicate the standardized variables such that $x_t = \frac{X_t}{\bar{Y}_t}$. The modified portfolio conditions are

$$\mathbb{E}_t \left[\beta_t \left(\frac{c_t}{c_{t+1}}\right)^{-\sigma} R^a_{t+1} e^{(1-\sigma)g}\right] = 1$$
$$\mathbb{E}_t \left[\beta_t \left(\frac{c^*_t}{c^*_{t+1}}\right)^{-\sigma} \left(\frac{\gamma_t}{\gamma_{t+1}}\right)^{-\sigma} R^a_{t+1} e^{(1-\sigma)g}\right] = 1$$

where the definition of Λ, $R^a_{t+1} = \frac{R^A_{t+1}}{e^g}$, and the assumption that $g = g^*$ have been used. Equivalent

[1] It is also important to remember that $\frac{\gamma_t}{\gamma_{t+1}}$ is just a function of the term of trade.

expressions apply for the liability portfolio conditions and R^l_{t+1}. The market clearing conditions are

$$\begin{aligned} y_t &= \left(\frac{P_{H,t}}{P_t}\right)^{-\sigma} \left[\lambda c_t + (1-\lambda)\gamma_t^{\phi} c_t^*\right] \\ \frac{Y_0^*}{Y_0} y_t^* &= \left(\frac{P_{F,t}}{P_t}\right)^{-\sigma} \left[(1-\lambda) c_t + \lambda \gamma_t^{\phi} c_t^*\right] \end{aligned}$$

where it is worth noticing that $y_t = e^{\varepsilon_t}$ and $y_t^* = e^{\varepsilon_t^*}$ and the initial values of output in the two countries are allowed to be different. Finally, the budget constraint becomes

$$b_t = R_t^m b_{t-1} - R_t^x b_{t-1}^m - (y_t - c_t) \tag{17}$$

where the definitions of R^m and R^x follow from those of the two return rates.

1.5 Steady State

All relative prices are assumed to be 1 in steady state. This implies $\gamma_{ss} = \tau_{ss} = 1$ and the PPP holds. I use the subscript ss to indicate the steady state value of a variable. As a consequence, the relative demands of subvarieties H and F are

$$\begin{aligned} c_{H,ss} &= \lambda c_{ss} \\ c_{F,ss} &= (1-\lambda) c_{ss} \end{aligned}$$

with the symmetric relations for $c^*_{H,t}$ and $c^*_{F,t}$. The steady state technological level is 1, as implied by $\varepsilon_{ss} = 0$, and also $y_{ss} = 1$ then. From the portfolio first order conditions, we obtain the steady state values of the return rates, which is going to be the same for both of them

$$R^a_{ss} = R^l_{ss} = R^m_{ss} = \frac{e^{(\sigma-1)g}}{\beta_{ss}}$$

This implies a zero steady state for excess returns as well $R^x_{ss} = 0$.

The linear approximation of the model is taken around a steady state in which the net debt position is not necessarily assumed to be zero. From the budget constraint (17), we derive the steady state relation between consumption and debt for a given net debt b_{ss}

$$c_{ss} = 1 + b_{ss}(1 - R^m_{ss}) \tag{18}$$

At the same time, the two market clearing conditions are

$$1 = \lambda c_{ss} + (1-\lambda)\, c^*_{ss} \tag{19}$$

$$\frac{Y_0^*}{Y_0} = (1-\lambda)\, c_{ss} + \lambda c^*_{ss} \tag{20}$$

Equation (19) implies $c^*_{ss} = \frac{1-\lambda c_{ss}}{1-\lambda}$; substituting for this into (20), we obtain

$$\frac{Y_0^*}{Y_0} = \frac{\lambda}{1-\lambda} + \frac{1-2\lambda}{1-\lambda} c_{ss}$$

When $b_{ss} = 0$, $c_{ss} = 1$ and the steady state ratio of the sizes of the two economies is 1 and we have a perfectly symmetric model. The same result would occur if there is no home bias in consumption and $\lambda = \frac{1}{2}$, independently of the value of b_{ss}. In order to have both features in this model, we need to have $Y_0^* \neq Y_0$; however, for typical calibrations, the two countries will have comparable sizes and they would both satisfy the large-country assumption.

1.6 Solution, Calibration, and Simulation

The solution of the model is obtained by standard approximation methods for a linearized first order version of the model around the steady state defined above. I adopt the first step of the strategy proposed by Devereux and Sutherland (2011) to deal with undetermined portfolio holdings when a first order approximation is taken, which allows to solve for the dynamics of b_t independently of the gross asset positions. Portfolio holdings can then be found applying the second step of this solution strategy. This approach recognizes that the first order linearization of $R^x_t b^m_{t-1}$ in (17) is simply an *i.i.d.* component and treat it equivalently to an extra exogenous shock in the model. The linearized budget constraint reads

$$\hat{b}_t = R^m_{ss}\left(\hat{R}^m_t + \hat{b}_{t-1}\right) - z_t - \frac{1}{b_{ss}}\left(\hat{y}_t - c_{ss}\hat{c}_t\right) \tag{21}$$

where $\hat{x}$ represent the log-deviation of a variable x from steady state and $\hat{R}^m_t = \frac{1}{2}\left(\hat{R}^a_t + \hat{R}^l_t\right)$. The term z_t corresponds to the linearization of $R^x_t b^m_{t-1}$, which is simply $z_t = b^m_{ss}\left(\hat{R}^a_t - \hat{R}^l_t\right) = b^m_{ss}\hat{R}^x_t$ because $R^x_{ss} = 0$. Given this step a solution of the model is straightforward.[2]

The benchmark calibration of the model relies on a very standard set of parameter values. The subjective intertemporal discount factor is set to $\beta_{ss} = .99$ and the gross growth rate to $g = 1.0025$; assuming that one period in the model corresponds to a quarter, these values imply that the annual discount factor is .96

[2] The solution exploits the property $\mathbb{E}_t\left(\hat{R}^a_{t+1} - \hat{R}^l_{t+1}\right) = 0$ from which $\hat{R}^x$ can be substitute by an *i.i.d.* process. The solution of the model is then computed in Matlab using the Dynare suite.

and the annual growth rate is 1%. The consumption home bias parameter λ is chosen to be 0.75 which ideally correspond to an import to GDP steady state ratio of 25%. ψ regulates the speed at which the net wealth reverts to its steady state value after a shock, it must be positive and small so that this assumption of stationarity does not affect the short run dynamics of the model, I pick $\psi = .01$. The CRRA coefficient in the utility function is set to the very standard value of $\sigma = 2$ and the elasticity of substitution between H and F goods ϕ is .9. This value falls in the lower end of the empirical estimates and it is preferred because it allows for larger movements of the exchange rate that are not sufficiently generated by the other components of the model. Finally, I target a debt to GDP ratio $b_{ss} = 4$. Table 1 summarizes the picks in this calibration.

Calibration	
Steady state net debt	$b_{ss} = 4$
Discount factor	$\beta_{ss} = .99$
Growth rate	$g = .0025$
CRRA risk aversion coefficient	$\sigma = 2$
Consumption home bias	$\lambda = .75$
Goods subst. elasticity	$\phi = .9$
β responsiveness to cons.	$\psi = .01$
Shocks:	
Autoregressive coefficients	$\rho_\varepsilon = .8$
	$\rho_\beta = .7$
std productivity shock	$v_\varepsilon = .0007$
std demand shock	$v_u = .0008$
std debt shock	$v_z = .004$

Table 1: Calibration of the model.

This calibration produces a steady state value for the quarterly gross returns of $R^a_{ss} = R^l_{ss} = 1.0125$, which corresponds to a 5% annual return. The steady states domestic consumption to output trend ratio is $c_{ss} = .95$, while the foreign ratio is $c^*_{ss} = 1.15$. Being a net debtor in steady state, the H country needs to run a steady state trade surplus; this surplus is 5% of GDP. This stylized model does not include capital investment and government spending, which makes the entire trade balance determined only by the consumption dynamics. This surplus is matched by the deficit of the F country, which implies a ratio of the two production levels $Y^*_0/Y_0 = 1.1$.

In addition to this very standard core calibration, the shock processes are selected in order to broadly reflect some of the moments of the US dataset analyzed in the paper. In particular, I focus on the relative volatilities of the variables included in the VAR models with respect to the volatility of the external debt and the correlation of debt and returns. The productivity shocks are assumed to be slightly more persistent than the consumption shocks, so we set $\rho_\varepsilon = \rho^*_\varepsilon = .8$ and $\rho_\beta = \rho^*_\beta = .7$. The linearization of the budget constraint in (21) also introduces an additional innovation that is used to enrich the overall dynamics of

A: Relative Standard Deviations

	y	c	d	b	γ	R^M	R^X	R^F
model	.029	.049	.040	1	.035	.054	.002	.021
data	.032	.062	.035	1	.057	.067	.045	.017

B: Correlations with debt

	R^M	R^X	R^F
model	.119	$-.101$	$-.202$
data	$-.011$	$-.056$	$-.105$

Table 2: Panel A: matching the relative volatility of the variables of the model with respect to debt. Panel B: correlations with debt.

the model. The equation standard errors of a VAR estimated by OLS with the US data are used to get a sense of the relative magnitude of the innovation to GDP, consumption and debt. The first two have similar equations standard errors, while that of the debt equation is a factor 10 bigger.[3] With this information in mind, the standard deviations of the productivity and demand shocks are set to $v_\varepsilon = .0007$ and $v_u = .0008$ respectively; the standard error of z is set to $v_z = .004$.

Table 2 illustrates the performance of the calibrated model in terms of relative volatility of the variables with respect to debt and the correlations with debt. The moments are computed from a $1,000$ periods sample taken a the end of a $20,000$ periods simulation with innovations drawn from a joint normal distribution with the standard deviations defined by v_ε, v_u, and v_z. The baseline calibration already allows the model to replicate the relative volatilities in a satisfactory way, except for the excess returns and, in part, the real exchange rate. The model generates a very small standard deviation compared to the data for R^X; for this reason, it is necessary to introduce larger deviations to make the effects of detrending clear in the VAR of Section 4.3 of the paper. On the other hand, a small ϕ is sufficient to compensate at least in part for the lower volatility of γ. The contemporaneous correlations are less satisfactory, especially for R^M which also has the wrong sign. The model generates stronger negative correlations between the other two returns and the net debt. The model fails matching the correlations with consumption and trade balance (not reported) due to the large persistent trade deficit of the US during a period of long-term growth of the debt position of the country. This model instead is based on the necessary condition that a steady state net debt has to be supported by positive trade surpluses, which is not found in the data in the post-1970 sample.

Figure 1 illustrates the fluctuations around their trends of the simulated series of output and consumption (top panel) along with excess and average returns. The fluctuations of the two series are comparable and consumption often crosses the output line causing a switch in the sign of the trade balance. In addition to

[3] A VAR with the usual 6 variables of the baseline specification of the paper, the consumption to GDP ratio and the GDP cycle around an HP-filtered trend is estimated. The equation standard errors for GDP, consumption, and debt are rispectively .007, .004, and .044.

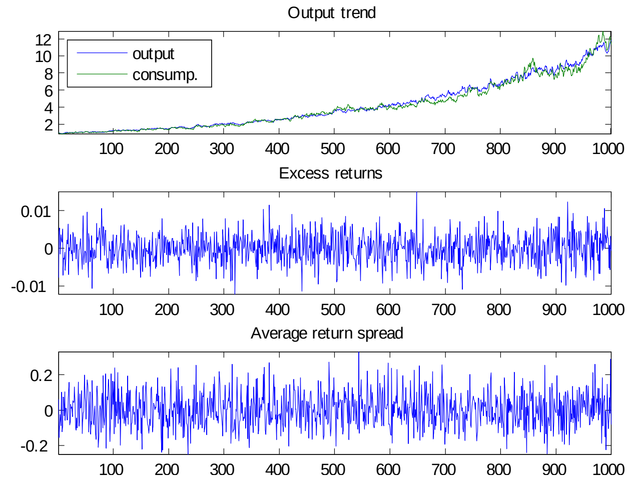

Figure 1: Simulated data for baseline calibration. Top panel: Output. Bottom panels: Excess and Average Returns. Simulation sample: 1000 periods.

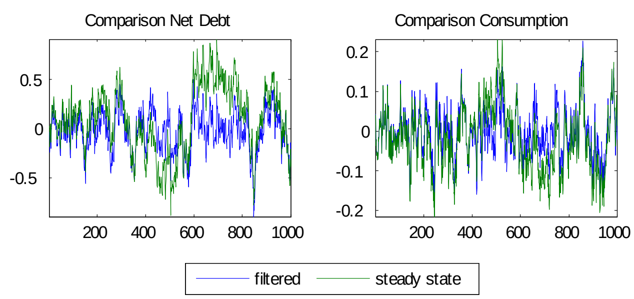

Figure 2: Comparison between filtered deviations and deviations from the theoretical steady state. Simulated data for baseline calibration. Left panel: Net debt. Right panel: Consumption.

that, the two bottom panels show the simulated R^X and $R^M - R^F$. Both the excess returns and the average return spread display the expected $i.i.d.$ type behavior.

Figure 2 shows the effects of filtering the data to remove the trend for the net debt and consumption. The figure compares the detrended series after the HP-filter is applied to the simulated data and the model-based deviations from the steady state implied by the balanced growth path assumption necessary to linearize the model. Even with the series generated by the baseline simulation, we can see that during periods of more persistent deviations from the steady state, due to occasional sequences of similar shocks, filtering generates much smoother cycle components. This is very clear, for instance, for the debt between period $600-800$, but the same is true also for consumption although the difference is less sharp. This difference in the outcome of the filtering procedure is the underlying source that explains the spurious results about the predictability of the excess returns in Gourinchas and Rey. The impulse response functions of non-detrended and detrended VAR using this data are reported in Section 3.3.

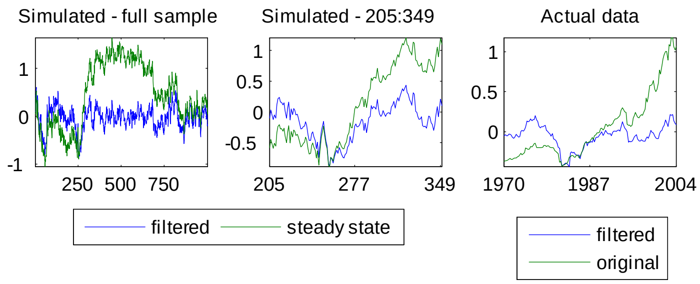

Figure 3: Comparison between filtered deviations and deviations from the theoretical steady state. Simulated data for second scenario. Left panel: Full sample. Central panel: blow-up for the period 205:349. Right panel: actual data - HP-filtered deviations Vs debt to GDP ratio.

In the data used to obtain the VAR results presented in Section 4.3, a sequence of negative demand shocks to β_t is mechanically imposed during the simulation in order to increase the persistence of the deviations from the steady state and amplify the size of the effects described in the previous paragraph. This second simulation is run for $10,000$ periods, but at period $5,000$ the shock to domestic preferences, u_t, is set to $-.01$ and it assumed to be linearly re-absorbed for the following 500 periods. Finally, a sample of 1000 periods centered around this block is selected for the VAR analysis presented in Figure 5. The two panels on the left hand side of Figure 3 compare HP-filtered and theoretical deviations for the debt variable in this second case; the last panel reports the conceptually equivalent comparison between the debt to GDP ratio and HP-filtered deviations for the actual Gourinchas and Rey US dataset.[4] The middle panel of the figure is a blow-up of the left panel for the sub-sample 205:349; this panel shows that the shock sequence of the second simulation produces a set of effects on the series qualitative comparable to what observed in the actual data.

2 Additional Approximation Results

2.1 Linearization à la Gourinchas and Rey

The data used in the detrended VAR model are treated following Gourinchas and Rey (2007) in order to guarantee the best comparability between my framework and their results. This section, which largely borrows from their work, shows how a constraint linearized following their approach is equivalent to the linearization of the budget constraint in equation (5) of the paper taken around a time invariant steady state. In order to preserve the logic of a balance growth path economy assumed in Gourinchas and Rey, I will linearize the constraint standardized by net worth in equation (9) of the paper and copied below for convenience. This choice is made for practical purposes only, since it does not bear any conceptual

[4] For graphical rendering, the filtered series is re-scaled in order to match the trough of the original ratio.

implication for the derivation, but facilitates the comparison with Gourinchas and Rey

$$l_t - a_t = R^l_t l_{t-1} - R^a_t a_{t-1} - (x_t - m_t) \tag{22}$$

This constraint can be approximated around a deterministic economy in which a trend is allowed instead of a fixed steady state point as done in this paper since l_t, a_t, x_t, and m_t clearly show a time trend in the data. The final form of the linearization, under the assumption of Gourinchas and Rey, are equivalent. Let the counterpart of $z_t \in \{l_t\ a_t\ x_t\ m_t\}$ in the deterministic economy be denoted by z^{ss}_t. These variables allow one to approximate for the deterministic time trend of the corresponding stochastic variables. On the other hand, R^a_t and R^l_t are assumed to be stationary and to have a constant steady state deterministic counterpart. The steady state level of the two returns, R^{ss}, is the same and from the equivalent of conditions (A1) in the Appendix of the paper it is found to be $\tilde{\beta}^{-1} = \beta^{-1}\Gamma^{-1}$ where Γ is the steady state growth rate of household net worth. The constraint in the deterministic economy can be written as

$$l^{ss}_t - a^{ss}_t = \tilde{\beta}^{-1}\left(l^{ss}_{t-1} - a^{ss}_{t-1}\right) - (x^{ss}_t - m^{ss}_t) \tag{23}$$

The deterministic economy is assumed to asymptotically settle along a balanced-growth path. This assumption implies that l_t, a_t, x_t, and m_t share a common deterministic trend and that the growth rates of L_t, A_t, X_t, and M_t converge to that of net worth.[5] A common trend can be expressed as $z^{ss}_t = \bar{z}\gamma_t$ for all $z_t \in \{l_t\ a_t\ x_t\ m_t\}$. The balanced-growth path implies $\lim_{t\to\infty} \gamma_t = 1$. If $z^{ss}_t = \bar{z}\gamma_t$, then the following ratios that are used as weights in the log-linearization of the constraint are constant

$$\mu^z_t = \frac{z^{ss}_t}{l^{ss}_t - a^{ss}_t} = \frac{\bar{z}}{\bar{l} - \bar{a}}$$

and constraint (23) asymptotically is equivalent to

$$\bar{l} - \bar{a} = \tilde{\beta}^{-1}\left(\bar{l} - \bar{a}\right) - (\bar{x} - \bar{m}) \tag{24}$$

From (24), it is easy to see that

$$\tilde{\beta} = \frac{1}{1 + \frac{\bar{x}-\bar{m}}{\bar{l}-\bar{a}}}$$

A forward looking solution for $\hat{b}$ in (26) below requires $\tilde{\beta} < 1$, which holds if the condition $\frac{\bar{x}-\bar{m}}{\bar{l}-\bar{a}} > 0$ is satisfied. The trade balance and the net debt position must have the same sign, which occurs if either a long

[5] This point is made by Assumption 2 and 3 in Gourinchas and Rey (2007).

run net debt position is supported by a trade surplus or a positive net asset position by a deficit.

Letting $\hat{z}_t$ indicate the log-deviations of z_t from z_t^{ss} and, in the same fashion, letting $\hat{R}_t^i$ for $i = a, l$ be the deviation of the return rates from their steady state value, a log-linearization of (22) around (23) becomes

$$l_t^{ss}\hat{l}_t - a_t^{ss}\hat{a}_t = \tilde{\beta}^{-1}\left\{l_{t-1}^{ss}\left(\hat{R}_t^l + \hat{l}_{t-1}\right) - a_{t-1}^{ss}\left(\hat{R}_t^a + \hat{a}_{t-1}\right)\right\} - \left(x_t^{ss}\hat{x}_t - m_t^{ss}\hat{m}_t\right) \tag{25}$$

Dividing (25) through by $l_t^{ss} - a_t^{ss}$, remembering that under the aforementioned assumptions the weights are constant and $l_{t-1}^{ss} - a_{t-1}^{ss}$ converges to $l_t^{ss} - a_t^{ss}$, it is easy to obtain

$$\left|\mu^l\right|\hat{l}_t - \left|\mu^a\right|\hat{a}_t = \tilde{\beta}^{-1}\left\{\left|\mu^l\right|\left(\hat{R}_t^l + \hat{l}_{t-1}\right) - \left|\mu^a\right|\left(\hat{R}_t^a + \hat{a}_{t-1}\right)\right\} - \left(\left|\mu^x\right|\hat{x}_t - \left|\mu^m\right|\hat{m}_t\right)$$

Finally, let us use the definitions of net debt in deviation from the deterministic trend $\hat{b}_t = \left|\mu^l\right|\hat{l}_t - \left|\mu^a\right|\hat{a}_t$, the deviation trade balance $\hat{d}_t = \left|\mu^x\right|\hat{x}_t - \left|\mu^m\right|\hat{m}_t$, and the difference in returns in this linearized context $\hat{R}_t^* = \left|\mu^a\right|\hat{R}_t^a - \left|\mu^l\right|\hat{R}_t^l$ to get

$$\hat{b}_t = \tilde{\beta}^{-1}\left[\hat{b}_{t-1} - \hat{R}_t^*\right] - \hat{d}_t \tag{26}$$

The absolute values in the weights are necessary to deal at the same time with the possibility of $l_t^{ss} - a_t^{ss}$ being positive or negative, which is not relevant in this paper since the weights are not re-scaled by the net position anymore. When $l_t^{ss} - a_t^{ss} < 0$, for instance, a positive deviation of $\hat{l}_t$ in $\mu^l\hat{l}_t - \mu^a\hat{a}_t$ would decrease rather than increase net liabilities. For this reason, the absolute values must be introduced if we want to jointly represent the two cases.

Equation (26) can be solved forward for $\hat{b}_{t-1}$ if $\tilde{\beta} < 1$. After imposing the transversality condition $\lim_{T\to\infty}\tilde{\beta}^T\hat{b}_{t+T} = 0$ and using the time t equation again, we obtain (27) which is clearly equivalent to equation (5) reported in Section 2.3 of the paper

$$\hat{b}_{t-1} = \mathbb{E}_t\sum_{i=1}^{\infty}\tilde{\beta}^{i-1}\left(\hat{R}_{t+i}^* + \tilde{\beta}\hat{d}_{t+i}\right) \tag{27}$$

2.2 Second order approximation of the external constraint

In this Appendix, I briefly discuss the second order approximation to the budget constraint and show how the theoretical results of the paper can be extended to higher order of approximation with only a few differences due to the different characterization of the expected excess returns. First, let us introduce with a little abuse of notation a few useful definition to make the following exposition easier

$$
\begin{aligned}
\hat{B}_t + \frac{1}{2}\hat{B}_t^2 &= L_{ss}\left(\hat{L}_t + \frac{1}{2}\hat{L}_t^2\right) - A_{ss}\left(\hat{A}_t + \frac{1}{2}\hat{A}_t^2\right) \\
\hat{D}_t + \frac{1}{2}\hat{D}_t^2 &= X_{ss}\left(\hat{X}_t + \frac{1}{2}\hat{X}_t^2\right) - M_{ss}\left(\hat{M}_t + \frac{1}{2}\hat{M}_t^2\right) \\
\hat{R}_t^* + \frac{1}{2}\left(\hat{R}_t^*\right)^2 &= A_{ss}\left(\hat{R}_t^A + \frac{1}{2}\left(\hat{R}_t^A\right)^2\right) - L_{ss}\left(\hat{R}_t^L + \frac{1}{2}\left(\hat{R}_t^L\right)^2\right) \\
\hat{R}_t^X + \frac{1}{2}\left(\hat{R}_t^X\right)^2 &= \left(\hat{R}_t^A - \hat{R}_t^L\right) + \frac{1}{2}\left[\left(\hat{R}_t^A\right)^2 - \left(\hat{R}_t^L\right)^2\right] \\
\hat{R}_t^M + \frac{1}{2}\left(\hat{R}_t^M\right)^2 &= \frac{\hat{R}_t^A + \hat{R}_t^L}{2} + \frac{1}{2}\left[\frac{\left(\hat{R}_t^A\right)^2 + \left(\hat{R}_t^L\right)^2}{2}\right] \\
\hat{\phi}_t &= A_{ss}\hat{A}_{t-1}\hat{R}_t^A - L_{ss}\hat{L}_{t-1}\hat{R}_t^L
\end{aligned}
$$

I start with the linearization of the version of the budget constraint in period t in equation (1) of the paper

$$
\hat{B}_t + \frac{1}{2}\hat{B}_t^2 = \beta^{-1}\left(\hat{B}_{t-1} + \frac{1}{2}\hat{B}_{t-1}^2 - \hat{R}_t^* - \frac{1}{2}\left(\hat{R}_t^*\right)^2 - \hat{\phi}_t\right) - \hat{D}_t - \frac{1}{2}\hat{D}_t^2 \tag{28}
$$

Solving this equation forward for $\hat{B}_{t-1} + \frac{1}{2}\hat{B}_{t-1}^2$, applying a suitable transversality condition, and using once again the linearized constraint at period t, the solution to the intertemporal linearized constraint up to the second order of approximation is

$$
\hat{B}_t + \frac{1}{2}\hat{B}_t^2 = \mathbb{E}_t\sum_{i=1}^{\infty}\beta^{i-1}\left[\left(\hat{R}_{t+i}^* + \frac{1}{2}\left(\hat{R}_{t+i}^*\right)^2 + \hat{\phi}_{t+i}\right) + \beta\left(\hat{D}_{t+i} + \frac{1}{2}\hat{D}_{t+i}^2\right)\right] \tag{29}
$$

It is not difficult to derive a few other results involving the second order approximations of excess and average returns. First, the well-known result about expected excess returns from the linearization of the portfolio conditions (A1) in the Appendix of the paper

$$
\mathbb{E}_t\left[\hat{R}_{t+1}^X + \frac{1}{2}\left(\hat{R}_{t+1}^X\right)^2 + \hat{\Lambda}_{t,t+1}\hat{R}_{t+1}^X\right] = 0 \tag{30}
$$

which shows that $\mathbb{E}_t R_{t+1}^X \neq 0$ up to a second order approximation. Second, an equivalent result for the average return is

$$
\mathbb{E}_t\left[\hat{\Lambda}_{t,t+1} + \frac{1}{2}\hat{\Lambda}_{t,t+1} + \hat{R}_{t+1}^M + \frac{1}{2}\left(\hat{R}_{t+1}^M\right)^2 + \hat{\Lambda}_{t,t+1}\hat{R}_{t+1}^M\right] = 0 \tag{31}
$$

Third, following the same steps as for the first order approximation, the two terms $\hat{R}_t^* + \frac{1}{2}\left(\hat{R}_t^*\right)^2$ and $\hat{\phi}_t$ in the summation can be re-written as

$$\begin{aligned}\hat{R}_t^* + \frac{1}{2}\left(\hat{R}_t^*\right)^2 &= B_{ss}^M\left[\hat{R}_t^X + \frac{1}{2}\left(\hat{R}_t^X\right)^2\right] - B_{ss}\left[\hat{R}_t^M + \frac{1}{2}\left(\hat{R}_t^M\right)^2\right] \\ \hat{\phi}_t &= \hat{B}_{t-1}^M \hat{R}_t^X - \hat{B}_{t-1}\hat{R}_t^M\end{aligned}$$

where $\hat{B}_t = L_{ss}\hat{L}_t - A_{ss}\hat{A}_t$ was introduced above and we define $\hat{B}_t^M = \frac{1}{2}\left(L_{ss}\hat{L}_t + A_{ss}\hat{A}_t\right)$. The expected value of these terms together simply becomes

$$\mathbb{E}_t\left[\hat{R}_{t+1}^* + \frac{1}{2}\left(\hat{R}_{t+1}^*\right)^2 + \hat{\phi}_{t+1}\right] = \mathbb{E}_t\left[B_{ss}^M\left(\hat{R}_{t+1}^X + \frac{1}{2}\left(\hat{R}_{t+1}^X\right)^2\right) - B_{ss}\left(\hat{R}_{t+1}^M + \frac{1}{2}\left(\hat{R}_{t+1}^M\right)^2\right) - \hat{B}_t\hat{R}_{t+1}^M\right] \tag{32}$$

where $\mathbb{E}_t\hat{R}_{t+1}^X = 0$ is used in $\hat{\phi}_t$, since $\hat{R}_{t+1}^X$ is only a first order term in the second order cross-product $\hat{B}_t^M\hat{R}_{t+1}^X$. This last equation shows that the effects of the excess return channel on the sustainability of the external debt in (29) would begin to be apparent at the second order of approximation.

Furthermore, using the approximated portfolio relations (30) and (31) and the first order approximation of the average return optimal condition $\mathbb{E}_t\hat{\Lambda}_{t,t+1} = -\mathbb{E}_t\hat{R}_{t+1}^M$ into equation (32), we obtain

$$\begin{aligned}\mathbb{E}_t\left[\hat{R}_{t+1}^* + \frac{1}{2}\left(\hat{R}_{t+1}^*\right)^2 + \hat{\phi}_{t+1}\right] &= B_{ss}^M\mathbb{E}_t\hat{\Lambda}_{t,t+1}\hat{R}_{t+1}^X + B_{ss}\mathbb{E}_t\hat{\Lambda}_{t,t+1}\hat{R}_{t+1}^M \\ &\quad + B_{ss}\mathbb{E}_t\left[\hat{\Lambda}_{t,t+1} + \frac{1}{2}\hat{\Lambda}_{t,t+1}^2 +\right] - \hat{B}_t\mathbb{E}_t\hat{\Lambda}_{t,t+1}\end{aligned} \tag{33}$$

which provides a clear decomposition of the contributions of the different channels to the sustainability of debt. In addition to trade, the last term in in (29), the SDF plays a major role in (33); as seen in the first order approximation, this component is strictly tied to the movements of the average returns both in its first and second order components. The average return channel affects sustainability also through its second moment interaction with the SDF represented by $\mathbb{E}_t\hat{\Lambda}_{t,t+1}\hat{R}_{t+1}^M$, which is a second order effect; an analogous second-order second-moment effect of the excess return channel is finally represented by $\mathbb{E}_t\hat{\Lambda}_{t,t+1}\hat{R}_{t+1}^X$.

3 Additional Empirical Results

This section presents further results in support of the empirical analysis in Section 4 of the paper.

3.1 Extended Non-Detrended VAR Analysis

The goal of this section is to show that neither future excess returns nor average return spreads are predicted by the US debt per se; however, shocks to the debt induce a response of the expected stochastic discount

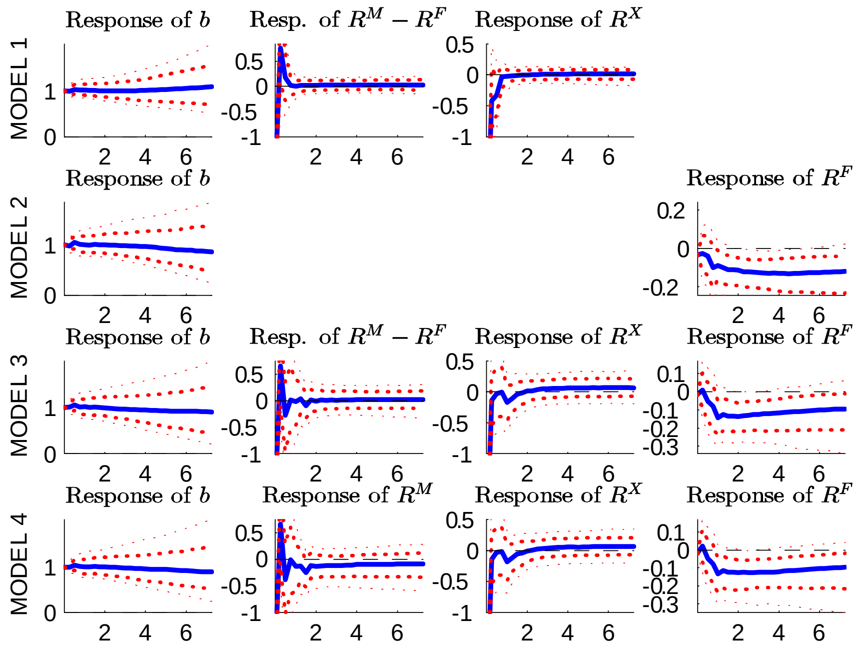

Figure 4: Comparison of the responses to a b unit shock in the non-detrended model. Model 1 specification $\left(d,\ b,\ \gamma,\ R^M - R^F,\ R^X\right)$; Model 2 $\left(d,\ b,\ \gamma,\ R^F\right)$; Model 3 $\left(d,\ b,\ \gamma,\ R^M - R^F,\ R^X,\ R^F\right)$; Model 4 $\left(d,\ b,\ \gamma,\ R^M,\ R^X,\ R^F\right)$. Band intervals correspond to the 14/86th and 5/95th percentiles. Years from the impulse on the x-axis. The ordering of the variables in the Cholesky decomposition is $\left(d,\ b,\ \gamma,\ R^M - R^F \text{ or } R^M,\ R^X\right)$.

factor consistent with the solvency of the intertemporal budget constraint. Only when the positive feedback effects of the SDF on the debt are taken into account, then some predictability of the returns is found too.

I follow an empirical strategy based on the comparison of the impulse response functions of four nested specifications of the same type of VAR model. Each specification is based on a selection of the following variables: the net foreign debt position b_t, the trade surplus d_t, the excess returns R_t^X, the depreciation rate of the real exchange rate γ_t, the three-month treasury bill rate R_t^F, the spread between average returns and risk-free rate $R_t^M - R_t^F$, or in alternative simply the average return rate R_t^M. The three-month US treasury bill rate is used as a proxy of the actual risk-free rate of the economy in order to capture the effects of the SDF on the intertemporal constraint. The structural shocks for the impulse response functions in each case are identified by a recursive Cholesky scheme based on the general ordering $\left(d,\ b,\ \gamma,\ R^M - R^F \text{ or } R^M,\ R^X,\ R^F\right)$, as in the main paper.

Figure 4 illustrates the response functions of b, $R^M - R^F$ (or R^M in the last model), R^X, and R^F to a unit impulse to b for the four specifications of the non-detrended model. In Model 1, five variables are considered $\left(d,\ b,\ \gamma,\ R^M - R^F,\ R^X\right)$. At impact, the excess returns display a very large drop, but then R^X promptly reverts toward zero and remains non significant after the third quarter.[6] Similarly, the response of the spread is large on impact and for the next couple of periods, but it is zero again after that. Finally, the shock to the debt is very persistent and the debt does not display any sign of mean reversion. The second specification replaces $R^M - R^F$ and R^X with the three-month treasury bill rate only $\left(d,\ b,\ \gamma,\ R^F \right)$. The objective of this modification is to show that the debt shock triggers a response of R^F consistent with a positive contribution of the SDF to the sustainability of debt. The risk-free rate decreases, and its response is statistically significant (the two pairs of dotted lines correspond to the $14/86th$ and $5/95th$ percentile intervals of the posterior distribution of the response functions computed by Monte Carlo integration). The introduction of information about the SDF changes also the response of b, which is still quite persistent but clearly mean-reverting now. The third specification puts together the full set of variables $\left(d,\ b,\ \gamma,\ R^M - R^F,\ R^X,\ R^F\right)$. Adding the two return components, $R^M - R^F$ and R^X, to the specification of Model 2 does not particularly affect the responses of R^F, while at the same time the mean-reversion of the response of b slightly improves. Part of the response of the excess returns is now positive and, although not strongly significant, the difference with the first specification is quite striking. On the other hand, the response of the average return spread remains extremely small and never significant. The last specification, in Model 4, replaces the spread with R^M in order to check whether the average returns mirrors the response of the risk-free rate as predicted by the theory for its first order component. This modification does not change the response of the other variables of the model and it shows how the zero response of the spread is explained by the similarity in the responses of these two variables.[7]

The key justification of the valuation channel is based on the dependence of the American foreign asset returns on the exchange rate. At impact, the real exchange rate appreciates followed by a long lasting, but not significant, depreciation; the excess return seems to follow the exchange rate response quite closely. Also the average return inherits its characteristics from the individual responses of asset and liability returns and, in this respect, a drop of the average return spread should be mostly driven by a fall in the liability return if the asset returns increase following the exchange rate depreciation. The relation between the responses of the exchange rate and those of the returns after the debt shock definitely deserves more attention. In Figure 5, I consider the separate response functions of R^a and R^l to a b shock in a VAR model similar to Model 3 and 4 in which excess and average returns are replaced by the asset and liability returns in order to evaluate their relative contribution to the two return channels. The first interesting observation is that the response

[6] The impact response is around -3. In the figure, the lower limit of the vertical axis is set to -1 in order to provide a better illustration of the response in the following periods too. A similar truncation of the y-axis is applied for the responses of R^M and $R^M - R^F$, which display similarly large falls on impact.

[7] As a further robustness check, I consider a specification in which consumption growth is added to Model 3 since the marginal utility of consumption would define the equilibrium SDF and it can improve the approximation of the expected discount factor. The response functions are reported in Section 4.1.

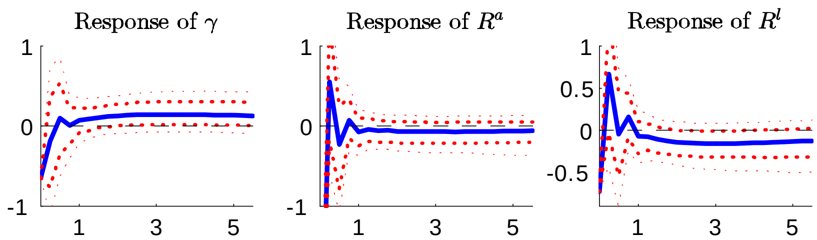

Figure 5: Responses of γ, R^a, and R^l to a b unit shock. The VAR uses specification 3 in which R^a and R^l replace and R^X and $R^M - R^F$. The limits of the vertical axes for the R^A and R^L responses are set to $(-1\ 1)$ and $(-1\ .9)$ respectively. Years from the impulse on the x-axis.

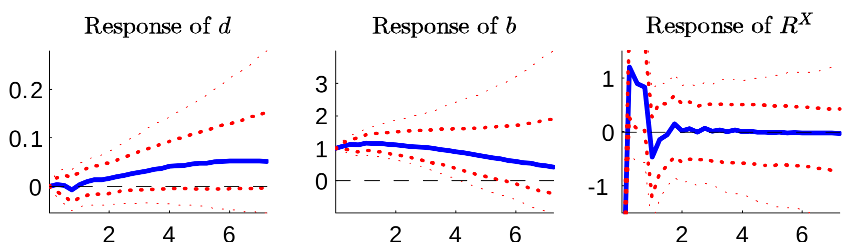

Figure 6: Response functions to a b unit shock for the shorter sample 1970:1-1990:1. Band intervals correspond to the 14/86th and 5/95th percentiles. Years from the impulse on the x-axis. The same specification as in Model 3 is estimated excluding the period of rapid growth of debt towards the end of the full sample.

of R^a follows the movements of the real exchange rate only at the very beginning of the response, and it is basically zero after the fourth quarter. The positive excess returns observed in Model 3 in the medium term is then mostly attributable to the response of R^l. This second effect cannot be directly attributed to the exchange rate valuation channel per se, even though it can be quite large. This kind of evidence supports the idea that the transmission of international shocks to the prices of assets can be very complex. Valuation effects related to the exchange rate are important determinants of the composition of the excess returns, because an exchange rate depreciation has a direct effect on R^a, but other less direct transmission channels can be at work too.

Finally, Figure 6 illustrates the responses of d, b, and R^X to a debt shock for the shorter sample 1970:1-1990:1 for Model 3, the main specification in the paper, following the discussion at the end of Section 4.1.

3.2 Extended Detrended VAR Analysis

The variance decomposition for the detrended model is reported in Figure 7. This can be directly compared to the decomposition of the non-detrended model in Section 4.1 of the paper. The effects of a shock to the trade balance are explored in Figure 8 instead. The debt $\hat{b}$ slightly decreases at the beginning and turns positive after two years. However the estimation bands are very large and the response is largely not

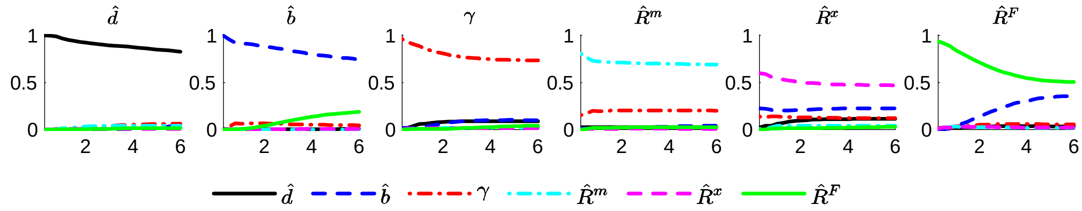

Figure 7: Variance decomposition for the main detrended model of the paper. Horizon of the forecast in years on the x-axis and baseline identification ordering.

significant. Given the budget constraint, a sequence of trade surpluses is normally expected to decrease future debt, but it is not clear from the figure.[8] The response of $\hat{R}^x$ to a $\hat{d}$ shock is large and significantly negative, while $\hat{R}^m$ is only marginally affected by this innovation. Furthermore, these shocks explain a portion of the variance of the excess returns comparable to that of the γ shocks in the variance decomposition, as reported in Figure 7. This effect was missing in the long-run model. Gourinchas and Rey's results suggest that the valuation effects have been restraining the growth of the US international debt, in face of insufficient trade balances; I find a sort of substitutability between $\hat{d}$ and $\hat{R}^x$ in the detrended model too, but this paper also shows that excess returns are systematically driving solvency over the long run.[9]

Canzoneri, Cumby, and Diba (2001) use a VAR framework analogous to mine to test for non-Ricardian regimes in the US government debt and deficit in the context of the fiscal theory of the price level. In this literature, the concept of Ricardian regime hinges on the role of the price index as a real deflator of the debt. The empirical methodology of this paper borrows from their approach in the use of the VAR and the impulse response functions to investigate the relationships among the variables of the model; however, a complete parallel with their analysis of the government debt is not feasible. At a first glance, the real exchange rate could seem a suitable candidate to mirror, in the external budget constraint of a country, the role that the price level has in the government budget constraint. However, my analysis shows that the valuation effects on the international debt rely on multiple factors and that the real exchange rate is simply one of them.

The spectral decomposition of the responses of the average returns, the excess returns, and the risk-

[8] These responses may obviously depend on the ordering of the Cholesky decomposition. Switching the relative position of debt and trade balance in order to constrain the response of the trade balance to be zero on impact makes the response of debt negative for the first two years after the shock.

[9] There is a clear similarity between the rebalancing mechanism outlined in Gourinchas and Rey (2007) and the concept of non-Ricardian regimes used in the monetary economics literature that studies the consolidate budget constraint of the government sector (see Sargent 1982). The intertemporal government budget constraint links monetary and fiscal policies and it requires that either taxes or seigniorage must be used to balance the government expenditure in present discounted value. A non-Ricardian regime is defined as a situation in which inflation and seigniorage adjust in order to ensure that the intertemporal budget constraint of the government is satisfied when taxes are not sufficient to achieve the goal. In this case, following the definitions by Leeper (1991), monetary policy is passive while fiscal policy is active and the regime is characterized by fiscal dominance.

In Gourinchas and Rey, excess returns play the same role for the international debt as seigniorage does for the government debt. A non-Ricardian regime in this framework can be defined as a situation of *trade dominance* in which excess returns systematically adjust to help the trade surpluses to maintain the balance of the intertemporal constraint.

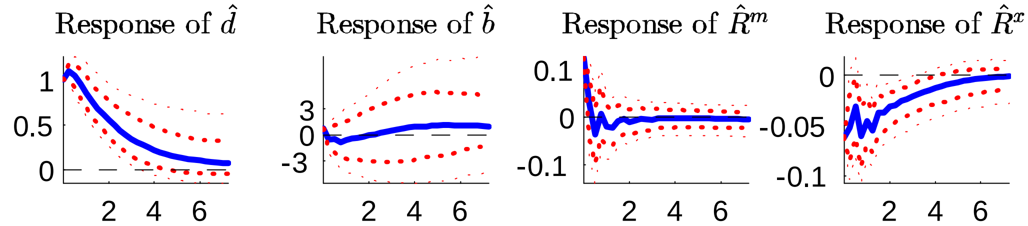

Figure 8: Response functions to a $\hat{d}$ unit shock. De-trended model - specification 4. Band intervals correspond to the 14/86th and 5/95th percentiles. Years from the impulse on the x-axis.

free rates to the debt shocks can help us to further understand the different characteristics of these three variables. Figure 9 illustrates the spectra of the responses in the two models where each spectral density has been standardized by the variance of the respective response function.[10,11] Two observations are noteworthy. The first one is that the spectral densities of the risk-free rate responses in both the models are highly concentrated around the low frequencies range $[-.2 \quad .2]$, the frequencies associated to periods greater than 32 quarters. Movements of the SDF linked to the debt mostly at low frequencies are consistent with the theoretical idea that sustainability is primarily a long-run phenomenon. On the contrary, the majority of the spectral densities of the responses of excess and average returns is distributed outside this set of frequencies. A second interesting observation is about the differences in the responses of average and excess returns in the two models. The spectra of these two variables in the detrended model have a relatively higher concentration in the range corresponding to business cycle frequencies, between .3 and 1.5, and display a drop at the very low frequency. Even though a similarly high relevance of the business cycle frequencies is found in the non-detrended model for the excess returns too, the detrending procedure has a significant impact on the spectrum of these variables at the very high frequency. As it can be seen in Figure 9, these high frequencies (3 quarters or less in period) are very important in the non-detrended model, becoming even the most important frequencies for the average return, but their contribution to the spectrum is largely reduced after detrending the data. In any case, also this evidence can be considered as broadly consistent with the weaker involvement of excess and average returns in the sustainability of the US debt described in

[10] Each spectral density in Figure 9 is smoothed using a Gaussian kernel. The point estimates of the spectrum for a given frequency are obtained as the discrete fourier transform of the sample autocovariances of the respective response function truncated at T lags. This allows to sample $2T-1$ frequencies, evenly spaced between $[-\pi, \ \pi]$. In order to increase the finesse of the smoothing, the spectrum is re-sampled at evenly spaced frequencies for n times, offsetting the first frequency each time by one nth of the distance between two consecutive frequencies. The $n(2T-1)$ point estimates of the spectrum are then averaged applying the normal weighting kernel in which use is made of the smoothing factor

$$\frac{\sqrt{n\,(2T-1)}}{2\pi h_0}$$

where the parameter h_0 controls the smoothness of the kernel. Figure 9 is based on the choice of parameters $T = 40$, $n = 3$, and $h_0 = 0.2$.

[11] These spectral densities can be seen as the gains of the fourth, fifth, and sixth components of the frequency responses to an impulse to the second component of the structural innovations vector.

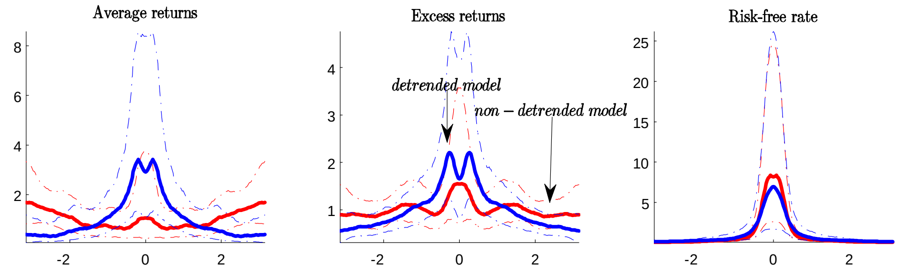

Figure 9: Spectra of the response of average returns, excess returns, and risk-free rate to a unit debt shock in the two models. Red lines refer to the non-detrended model and blue lines to the detrended model. The thick lines represent the median of the posterior distributions of the spectra; the dash-dot intervals correspond to the 14/86th percentile bands. Spectra are normalized by the variance of the response and computed using a normal Kernel (see footnote 26 for more details).

the rest of the empirical analysis of the paper.[12]

3.3 Extended Simulated-Data Analysis

In addition to the example used in Section 4.3 of the paper, based on the simulation characterized by the persistent deviations from the steady state mechanically introduced in the second simulation described in Section 1.6, I report the equivalent evidence for the first simulation with regular innovations and under the baseline calibration of the model in Figures 10-11. A two-lag VAR model is preferred in this case to the four-lag specification of the paper for graphical reason; the results hold also with four lags, but they would be simply delayed by a few periods. The shift in position and significance of the excess returns is preserved also with this simulation; the average return spread is now marginally significant, but only for two periods and reverts to zero very quickly and this effect is even smaller if four lags are used in the estimation. The response of the risk free rate is large and strongly significant as found in all the other cases, both empirically and theoretically. Overall the main results about the spurious predictability of excess returns are supported by this evidence.

Finally, Figures 12-13 illustrates the same set of responses for the second type of simulation. It is worth noticing how the average return spread response is now basically zero over the entire horizon , while the average return shows a negative and significant response for the first five periods. In all the cases I tried, the response of the average return was always negative and significant, but typically very short in time. The other responses share the same characteristic as in the previous set of figures. The trade balance response is

[12] The choice of the smoothing parameter h_0 may clearly affect the shape of these two spectral densities. The results are very similar for values of h_0 between 0.1 and 0.4. For values smaller than 0.1, which correspond to a very low degree of smoothing, the drop of the spectrum in the low frequency range is quite large too. For values greater than 0.5, the high smoothing makes the spectrum just flat in the low frequency range in both models. The same results are confirmed replacing the Gaussian with an Epanechnikov kernel and keeping the same smoothing factor. Since this kernel reduces the degree of smoothness for given h_0, the spectra display larger drops at low frequencies but their relative magnitude is preserved.

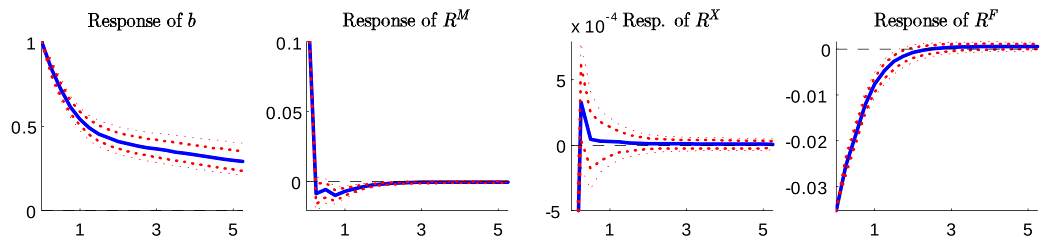

Figure 10: Response functions to a b unit shock. Non-detrended VAR model and simulated data based on the first simulation in Section 1.6. Band intervals correspond to the 14/86th and 5/95th percentiles. Years from the impulse on the x-axis. The variables in the model are the net foreign debt position b_t, the trade surplus d_t, the spread between average returns and three-month treasury bill rate $R^M - R^F$, the excess returns R_t^X, the depreciation rate of the real exchange rate γ_t, the risk free rate R_t^F. The vertical axis lower bound of the third panel is truncated to improve the visualization of the effects after the large impact response of R^X. The identification ordering is the same as in the main paper. Years from the impulse on the x-axis (4 periods in the simulation).

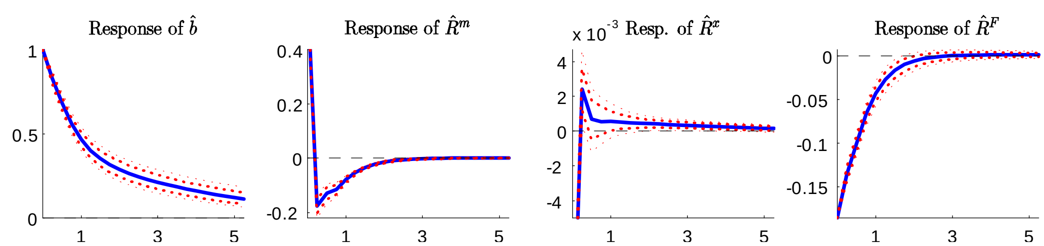

Figure 11: Response functions to a $\hat{b}$ unit shock. Detrended model and simulated data based on the first simulation in Section 1.6. The variables in this specification are net debt $\hat{b}_t$, trade surplus $\hat{d}_t$, depreciation rate of the real exchange rate γ_t, average return rate $\hat{R}_t^m$, excess returns $\hat{R}_t^x$, and of the real risk free rate $\hat{R}_t^F$. The identification ordering is the same as in the main paper. Band intervals correspond to the 14/86th and 5/95th percentiles. The vertical axis lower bound of the third panel is truncated to improve the visualization of the effects after the large impact response of $\hat{R}_t^x$. Years from the impulse on the x-axis (4 periods in the simulation).

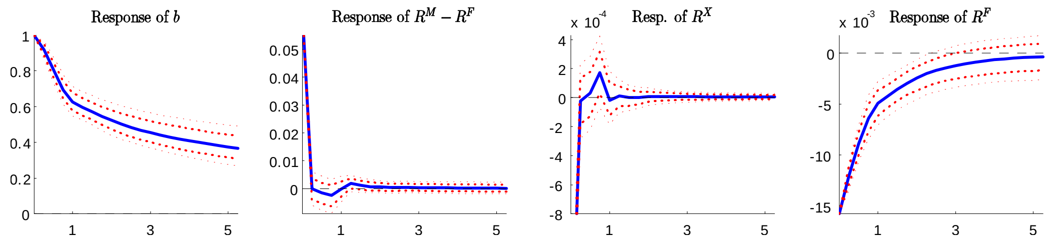

Figure 12: Response functions to a b unit shock. Detrended model and and simulated data based on the second simulation in Section 1.6, used also in Section 4.3 of the paper. Band intervals correspond to the 14/86th and 5/95th percentiles. Years from the impulse on the x-axis. The variables in the model are the net foreign debt position b_t, the trade surplus d_t, the spread between average returns and three-month treasury bill rate $R^M - R^F$, the excess returns R_t^X, the depreciation rate of the real exchange rate γ_t, the risk free rate R_t^F. The vertical axis lower bound of the third panel is truncated to improve the visualization of the effects after the large impact response of R^X. The identification ordering is the same as in the main paper. Years from the impulse on the x-axis (4 periods in the simulation).

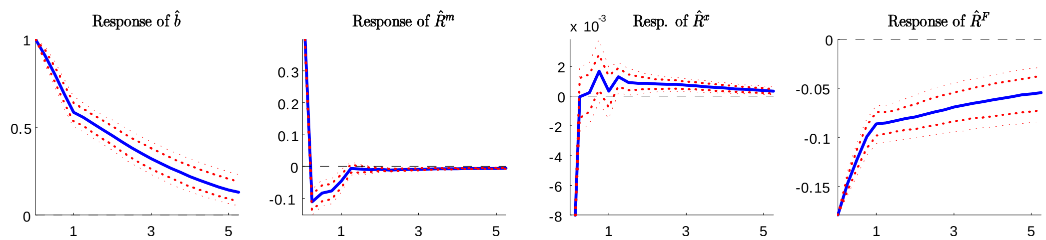

Figure 13: Response functions to a $\hat{b}$ unit shock. Detrended model and simulated data based on the second simulation in Section 1.6, used also in Section 4.3 of the paper. The variables in this specification are net debt $\hat{b}_t$, trade surplus $\hat{d}_t$, depreciation rate of the real exchange rate γ_t, average return rate $\hat{R}_t^m$, excess returns $\hat{R}_t^x$, and of the real risk free rate $\hat{R}_t^F$. The identification ordering is the same as in the main paper. Band intervals correspond to the 14/86th and 5/95th percentiles. The vertical axis lower bound of the third panel is truncated to improve the visualization of the effects after the large impact response of $\hat{R}_t^x$. Years from the impulse on the x-axis (4 periods in the simulation).

positive and very significant as expected from the theory for trade channel; on the other hand, the exchange rate exhibits a long lasting and significant appreciation, which is at odds with the empirical analysis. Both these responses are available form the author upon request.

4 Sensitivity Analysis

This section provides some extra detail on the overall suitability of the estimated VARs and of the prior parameters I use in the regressions.

4.1 Robustness of the Analysis

Comparing the baseline non-detrended VAR(4) model with VAR models with 2 and 6 lags, the basic nature of the responses remains the same for the majority of the variables. Typically, more lags make the initial portion of the impulse response functions more jagged, especially for γ and the returns, but the relatively stronger role of the risk-free rate is confirmed. Another important difference is that with two lags the excess returns remain negative for a a few periods after the shock, although being not significant, and it is then zero after that. The models with two or four lags are basically identical for the detrended VAR; increasing the number of lags reduces the fit of the model without modifying the overall outlook of the response functions.

An important point to discuss is the selection of the parameters for the Minnesota prior and the effects of different choices of the priors on the impulse response functions. The parameters used in the main results of the paper, $\phi_0 = .4$ and $\phi_1 = .5$, are quite standard. The choice of ϕ_1, the relative tightness parameter for the lags of the other variables of one equation, does not affect the results in a particular way once ϕ_0 is set. In the detrended VAR, the selection of ϕ_0 does not matter for the results either; therefore, ϕ_0 is studied in function of the behavior of the non-detrended VAR.

The results in the paper are obtained for $\phi_0 = .4$, which is in the middle range of feasible values. For looser priors, i.e. bigger values of ϕ_0, the same results are obtained. Considering smaller values of ϕ_0, around .1, increases the persistence of debt shock without having a strong impact on the other responses. Figure 14 illustrates the responses of $R^M - R^F$, R^X, and R^F to a b shock for a tighter prior corresponding to $\phi_0 = .1$ for model specification 3. Overall the conclusions are very robust to many possible parameterizations.

The effects of a different ordering of the variables in the Cholesky decomposition on the reciprocal responses of d and b are minimal for both detrended and non-detrended model. As known, the relative position of d and b does not matter for the responses of the subsequent variables of the VAR. The relative position of γ and R^X does not matter for the responses of the excess returns to b and d shocks either. The results for the detrended model are obtained under a specific choice of the smoothing parameter of the HP filter based on the Gourinchas and Rey (2007)'s approach. A more conventional smoothing parameter of the HP filter can be used to isolate the business cycle frequencies of the series. Also in this case, the overall effects remain qualitatively the same. The response functions are in general very significant, even though they are less persistent.[13]

Figure 15, instead, illustrates the responses of the returns and the debt to a unit debt shock in a version of the non-detrended model - specification 3 - in which consumption growth is added to the variables of the model. Consumption is related to the SDF definition through marginal utility and it can help in

[13] The plots for this last case are not reported here, but they are available from the author. The smoothing parameter of the HP filter is set to 1600.

empirically approximating the discount factor in the VAR model. The main differences in the results using the consumption specification of the model are the stronger mean-reversion of the debt shocks and a slightly less significant response of R^F compared to the non-detrended model in the paper. This specification is consistent with the evidence that links the sustainability of the US international debt position primarily to movements of the SDF.

4.2 Priors Selection, Forecast, and Overfitting

As discussed in Section 4.1, looser and tighter prior parameterizations of ϕ_0 return similar results in terms of impulse responses to the debt shocks. Here, I compare three values of $\phi_0 \in [.1 \;\; .4 \;\; .9]$ corresponding to a tight, an intermediate, and a loose prior respectively. The fit of the models under any of the three parameters is very similar in terms of forecast of the variables from the initial point. Figures 16 - 19 report the forecasts of the variables of the model from the initial point for the alternative choices of ϕ_0 in the non-detrended VAR model (specification 3) and for the main detrended model. The baseline specification does not show any evident over-fitting of the series, which means that the number of lags specified in the VAR regressions is not implausibly high. The problem of large initial transients considered by Sims (1996) is not an issue here. Therefore, it can be excluded that the root close to unit that characterizes the non-detrended VAR is artificially generated by the estimation procedure; the Minnesota prior is enough to correct for it. The main difference between the three priors is represented by the slightly larger bands on the forecasting of the debt for the intermediate prior.

From the companion form of the VAR under different specifications, I compute the eigenvalues of the system. The baseline non-detrended VAR (specification 3) has one eigenvalue close to unit, but still smaller, unit and the second largest eigenvalue definitely below 1: the values are .9987 and .9436. For specification 4 the two largest eigenvalues are .999 and .9444. The two largest eigenvalues of specification 2 of the non-detrended model are .9922 and .9524; while for specification 1 we obtain a largest root of the system bigger than one 1.0099. The baseline specification for the detrended VAR gives a strongly stationary system in which the largest eigenvalue is 0.9329 instead.

From the companion form, it is possible to reformulate the VAR model in the Jordan form by applying the Jordan decomposition of the companion matrix of the VAR. The new system is defined in a new set of variables, which are combinations of the variables in the companion form of the VAR (with weights given by the elements of the left eigenvectors of the companion matrix). In the Jordan form, the VAR in levels is reformulated as a set of 24 independent equations since there are 6 variables and 4 lags in the original VAR.

Let Z_t be the vector of the new variables in the Jordan form and Ω its covariance matrix. The new

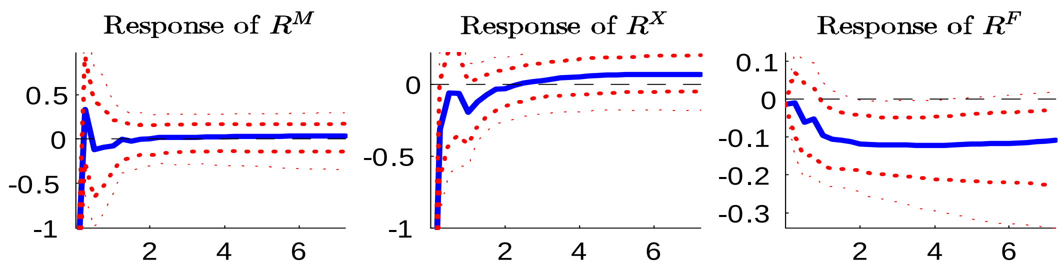

Figure 14: Response functions to a b unit shock. Non-detrended VAR model - specification 4 and tight prior for $\phi_1 = 10$. Band intervals correspond to the 14/86th and 5/95th percentiles. Years from the impulse on the x-axis. The limits of the vertical axes for the R^M and R^X responses are set to $(-1\ .75)$ and $(-1\ .5)$ respectively. Baseline identification ordering of the variables.

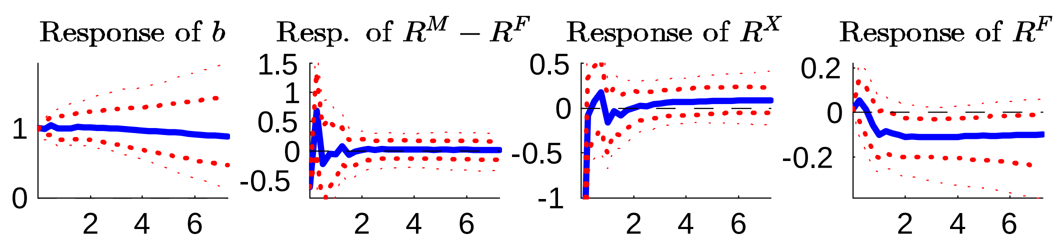

Figure 15: Response functions to a b unit shock in the model with consumption c. Non-detrended VAR model - specification $\left(c,\ d,\ b,\ \gamma,\ R^M - R^F,\ R^X,\ R^F\right)$. Band intervals correspond to the 14/86th and 5/95th percentiles. Years from the impulse on the x-axis. The limits of the vertical axes for the $R^M - R^F$ and R^X responses are set to $(-.8\ 1.5)$ and $(-1\ .5)$ respectively. Baseline identification ordering of the variables with c in first place.

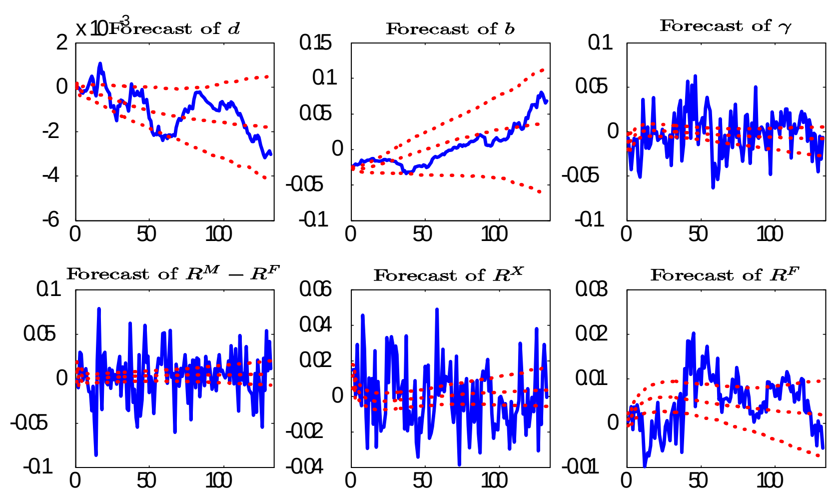

Figure 16: Initial point forecasts for the non-detrended VAR - specification 4 and $\phi_1 = 10^3$. Solid line actual, dotted line forecast. Band intervals correspond to the 14/86th percentiles.

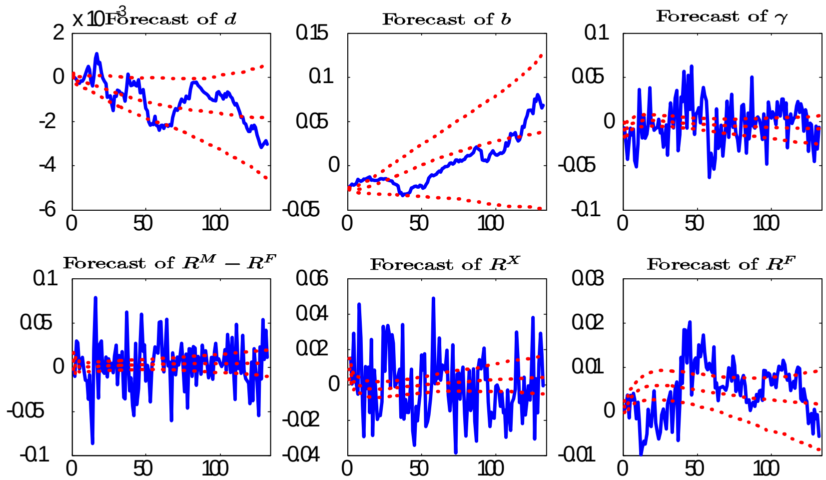

Figure 17: Initial point forecasts for the non-detrended VAR - specification 4 and $\phi_1 = 10$. Solid line actual, dotted line forecast. Band intervals correspond to the 14/86th percentiles.

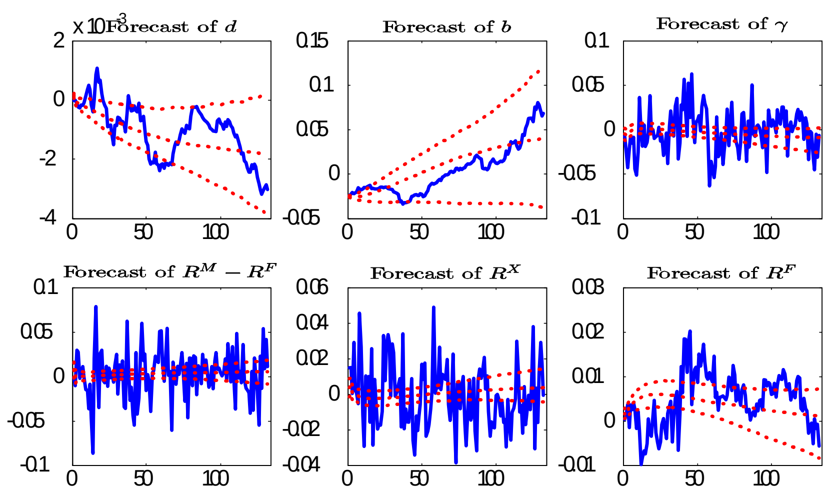

Figure 18: Initial point forecasts for the non-detrended VAR - specification 4 and $\phi_1 = .5$. Solid line actual, dotted line forecast. Band intervals correspond to the 14/86th percentiles

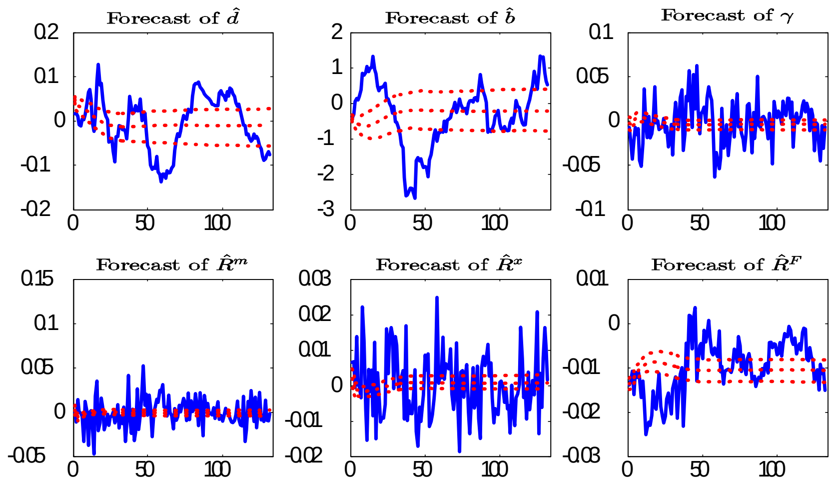

Figure 19: Initial point forecasts for the long-run VAR - specification 3 and loose prior for ϕ_1

variables in Z_t can be used to construct a Chi-squared test for the plausibility of the initial conditions of the data in relation to the distribution of Z_t implied by the posterior median estimates of the VAR model. In fact, the statistic $\tilde{Z}_t'\Omega\tilde{Z}_t$ (where $\tilde{Z}_t$ is the deviation of Z_t from its mean) must have a Chi-squared distribution with n degree of freedom, where n is the number of elements in Z_t.[14] If the test statistic evaluated at the initial point is too large, then we can conclude that the initial point is at odds with the distribution implied by the VAR estimates. At that point, one can decide how to use this information in order to evaluate the plausibility of the posterior estimates of the model in relation to the initial point. This can also be used in order to compare different models. Models with smaller $\tilde{Z}_0'\Omega\tilde{Z}_0$ may be preferred to models with very large statistics.

The Chi-squared test under the different specifications of the model does not reject the initial point for any of them relative to the fit of the rest of the sample, even though the fit slightly improves towards the end of the sample. Large differences are not found in the overall distribution of the other observations either. Among the four versions of the non-detrended VAR model, model 2 shows the highest fit, indirectly confirming the main contribution of the risk-free rate to the dynamics of the system relative to the other return channels.

[14] If the system has a non-stationary eigenvalue, the variance and mean of the corresponding element of Z_t are not well-defined any more. However, we can still construct the Chi-squared test for the remaining $n-1$ stationary components of Z_t.

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