Excess Returns, Average Returns, and the Adjustment Mechanism of the External Position of a Country

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Abstract

I provide a new decomposition of the external constraint of a country in which, in addition to trade and valuation channel, adjustments in the stochastic discount factor and the spread between average international returns and risk-free rate can offset a current debt position. The importance of these channels is empirically assessed using US data. A primary contribution of the discount factor and secondary effects of excess and average returns are found in the non-detrended analysis, confirming the theoretical characterization of the valuation effects in previous literature. By using detrended data instead, the role of excess returns would be spuriously overestimated.

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The large and persistent US current account deficit has raised many concerns about the sustainability of the American international debt position. As with any other borrower, the US is expected to generate positive savings at some point in the future in order to meet its liabilities. The standard conclusions of the intertemporal approach to the current account imply that those savings need to come from inverting the trade balance and becoming a net exporter. Since the early nineties, however, the size of the American current account deficit has been persistently growing, and a reversion of this trend occurred only after 2005. Nevertheless, at the time of this writing, the current account is still negative, and the current account deficit to GDP ratio is barely below 2%.

The Valuation Channel, introduced by Gourinchas and Rey (2007b), provides an appealing mechanism to reconcile the dynamics of the American current account and the lack of a vigorous rebalancing in trade with the intertemporal approach to the current account. Gourinchas and Rey find a systematic contribution of the valuation effects to the adjustment process of the American external imbalances. They study the US international imbalances in deviation from slow-moving deterministic trends and find empirical evidence that the valuation channel has largely complemented the conventional trade channel in eliminating these imbalances. However, the size of these effects is, to some extent, unexpected given the theoretical characterization of the excess returns provided by Tille and van Wincoop (2010) and Devereux and Sutherland (2010); this strand of the literature indicates that the valuation channel should only be of secondary importance for the overall sustainability of the intertemporal external constraint of a country being the valuation effects second order effects at best.

In this paper, the adjustment process of the international debt of a country is revisited in light of the differences between these empirical and theoretical results. The novelty of my approach is that the analysis is driven by the observation that a correct assessment of the external solvency of a country cannot be separated from the long-run dynamics of the debt itself. Adopting this perspective allows this research to make two important contributions. First, the paper explores a new theoretical decomposition of the intertemporal solution of the external constraint in which, in addition to trade surpluses and excess returns, the current debt can be offset by movements of the average returns on international securities as well as of the expected stochastic discount factor. Second, the paper examines the relative importance of these effects for the adjustments of the American international net debt position from an empirical perspective, and it shows crucial differences between the analysis including trends and the analysis in deviations from trends. In particular, neither the excess nor the average return channel is found to be significantly relevant for the solvency mechanism of the non-detrended American debt. Re-introducing trends overturns the conclusions of the importance of the valuation channel described by Gourinchas and Rey. On the contrary, the discount factor is closely linked to the rebalancing mechanism of the international position, and it exerts a reinforcing action on the other channels.

A correct assessment of the international solvency of a country must be based on the forward looking solution of the budget constraint, in which the utility-based stochastic discount factor is explicitly used. Since in general equilibrium the discount factor is linked to asset returns by the optimal portfolio conditions, I show that the solution of the intertemporal constraint entails four possible determinants of the current debt. The first determinant is the traditional trade channel represented by the expected future trade surpluses; the second is the Gourinchas and Rey excess returns on foreign assets, whose effects are amplified by the average gross holdings of international securities; the third determinant is the spread between the average returns on international securities and the risk-free rate, whose effects are magnified by the size of the net debt position of the country. I will refer to this component as the *average return channel*. Finally, these three components are discounted by the compounded risk-free rate in each period, which embeds the direct effects of the stochastic discount factor on the value of future flows. This decomposition is obtained without any degree of approximation of the budget constraint.

In spite of this decomposition, however, the two return channels must have only second order effects on debt relative to the trade channel and the stochastic discount factor. Similarly to Tille and van Wincoop (2010) and Devereux and Sutherland (2010), who illustrated this result for the excess returns, a linearization of the optimal portfolio conditions of the model shows that the spread between the average returns and the risk-free rate is zero in expectations up to a first order approximation. The empirical part of the paper tests the implications of the intertemporal budget constraint in the form derived here specifically for the US dataset used by Gourinchas and Rey (2007b), and it aims to reconcile the apparent contradiction between the empirical results in Gourinchas and Rey on the one hand and the theoretical observations of Tille and van Wincoop and Devereux and Sutherland on the other. Including in the analysis the average return channel reinforces the conclusions of the paper on the excess returns as well. The key result of the empirical exercise is showing that removing trends from the analysis, as done by Gourinchas and Rey, can spuriously amplify the relevance of this subset of secondary effects.

The empirical analysis is conducted using a VAR framework. The intertemporal budget constraint implies that an unexpected shock to the current level of external debt must be offset by an increase in the present discounted value of future streams of trade surpluses and excess returns or by a drop in the spread between average returns and risk-free rate. However, the responses of excess returns and spread are not expected to be significant. I assess these predictions comparing the responses of the four components of the budget constraint to an impulse to the foreign debt for two VAR models that differ in how data is treated: the first model uses detrended data as in Gourinchas and Rey (2007b), while the second model is estimated with non-detrended data. In the detrended model, positive and significant excess returns follow the initial debt shock; similarly, significant responses of the other components are found as well. This result reflects the evidence on the large valuation channel in Gourinchas and Rey. Three main differences are found with non-detrended data. First, the shock to the debt becomes clearly more persistent, while debt exhibits a quick mean reversion in the detrended VAR. Second, excess returns and average return spread respond only on impact to the debt shock, and the responses are negligible afterwards. This confirms the hypothesis that the two return components only have second order effects on the external constraint. Third, the significantly negative response to the shock of the risk-free rate is the principal source of adjustment of debt. The risk-free rate reflects the effects of the expected stochastic discount factor on the current debt. Furthermore, the average returns mirror the risk-free rate as predicted by the theory in both VARs.

As a final step, in order to explore the validity of these conclusions beyond the American case, I repeat the VAR analysis using simulated data from a general equilibrium open-economy model. The empirical evidence on the importance of the average returns is fully corroborated by the analysis with simulated data. In addition, the simulation also helps us gain a better understanding of the spurious results of the detrended model. The predictability of the excess returns in the VAR is the result of the use of the HP-filter on the simulated data in presence of sufficiently persistent deviations from the steady state of the model imposed by the balanced growth path assumption. Filtering data, even after a normalization, to remove trends can introduce distortions in the relations between the variables of the model conceptually equivalent to a miss-identification of the appropriate balanced growth path steady state.

The rest of the paper is organized as follows: Section 1 briefly discusses the relation of this paper to the literature. Section 2 outlines the open economy stochastic general equilibrium framework used as theoretical reference for the derivation of the intertemporal budget constraint. Section 3 describes the data for the empirical part, while Section 4 explains the empirical strategy and presents the results. Section 5, finally, concludes.

1 Relation to the Literature

The analysis in this paper does not provide any definitive conclusion about the actual sustainability of the American debt position. In order to answer this question, it would be necessary to observe a realization of the actual stochastic discount factor and to know how it co-moves with the trade balance and the other components of the budget constraint. Due to the difficulty of defining and quantifying the discount factor in empirical work, this is not a straightforward task. For this reason, a strand of literature on government debt sustainability has focused on providing and testing simpler sufficient conditions for sustainability. Trehan and Walsh (1991), Uctum and Wickens (1993), and Bohn (1998), in particular, check for endogenous responses of the budget surplus to the level of debt that automatically fulfill the transversality condition.[1] Even though the main concern of this paper is not testing for the international debt sustainability per se, I use the insights of this literature in a multivariate framework to study the effects of four components of the intertemporal budget constraint on debt.

Most of the literature has used deterministic discount factors in studying the transversality condition. In a stochastic environment, however, the choice of the discount factor cannot be arbitrary because it has to be endogenously determined by the model itself. As pointed out by Bohn (1995 and 1998), it is theoretically correct to derive the intertemporal budget constraint and the transversality condition using a consumption-based stochastic discount factor obtained from the optimality conditions of the representative household′s problem. This approach is crucial in order to derive the correct theoretical predictions that justify the empirical strategy of this paper.

Obstfeld and Rogoff (2001 and 2007) and Blanchard, Giavazzi, and Sa (2005) have provided the most prominent examples of the earliest literature on the sustainability of the American current account in which they focus on the rebalancing mechanisms expressed by the trade channel. In these models, a large real depreciation of the dollar triggers the shift of the international demand in real markets and the switch of sign of the trade balance that is necessary to drive the adjustment process of the current account. The fact that the depreciation of the domestic currency is the common denominator of both the trade and the valuation channel has aroused the interest of many economists and policy makers in the role of the valuation channel. Furthermore, the size of the valuation effects of exchange rate depreciations is proportional to the holdings of international assets of a country. As documented by Lane and Milesi-Ferreti (2004 and 2007), increasing global imbalances have been accompanied by increasing gross international positions and larger valuation effects for many countries. If this trend continues, the importance of the valuation channel may be destined to grow even more in the future as globalization increases.

Even though at first sight this paper might seem antithetical to Gourinchas and Rey (2007b), its correct place would be next to this work. Gourinchas and Rey impose a well-defined, balanced-growth path and the stationarity of the net assets to wealth ratios around this deterministic trend and, then, explore the theoretical relationships between net foreign asset position, future trade balances, and future asset returns.[2] My paper explicitly reintroduces trends in the analysis in order to show that they matter for the relation between sustainability, valuation and average return channel.

From a theoretical perspective, Tille and van Wincoop (2010) demonstrate that the role of excess returns

[1] The earlier literature on sustainability proposed tests based on the maintained assumption that the debt position is stationary or integrated of order one. See, among others, Flavin and Hamilton (1986), Wilcox (1989), and Trehan and Walsh (1988). This approach is usually considered less satisfactory because it simply imposes that the transversality condition is satisfied.

[2] This is the typical assumption in growth models and it makes the transversality condition satisfied by construction (see Bohn 2008).

in the solvency of the external imbalances is small and limited to higher order effects. Up to a first order, only trade balances can support the solvency of the intertemporal constraint of a country because expected excess returns are zero. Pavlova and Rigobon (2010) use a model with an equilibrium characterized by a closed-form solution to study the role of expected and unexpected capital gains on the dynamics of net foreign assets. They suggest that expected capital gains have a stabilizing effect on the trade balance, but the correlation between unexpected capital gains and trade balance is significantly smaller. Devereux and Sutherland (2010) show that the predictability of excess returns can be explained by constant, possibly non-zero, second order components or by time-varying third order components at best; the same conclusion is independently reached by Tille and van Wincoop (2010) using a different framework. Excess returns responses are smaller relative to those of other variables in my empirical VAR analysis; furthermore, I extend the same type of conclusion to the spread between average returns and the risk-free rate too.

This work is also complementary to the literature that describes the current imbalances as the outcome of equilibrium models. Even though they pursue an extremely different perspective on the problem, the papers in this area from Caballero, Farhi, and Gourinchas (2008) to Coeurdacier, Kollmann, and Martin (2010) draw their conclusions conditionally on the imposition of a long-run steady state.

Finally, Canzoneri, Cumby, and Diba (2001) use a two-variable VAR to study whether the fiscal theory of price level was supported by the US government debt and budget surplus data. I also use a VAR as an analytical tool to examine the relations among the variables of this model. Following the approach of their paper, the impulse response functions of the VAR are used to draw some conclusions about the responses of the trade balance, excess and average returns to unexpected shock to the external debt. The main purpose here, however, is to use these responses to elucidate the mechanisms that support the sustainability of the debt.

2 Theoretical Framework

Following the considerations of Bohn (1995), in this paper I study the dynamics of the foreign asset position of a country in the context of an open economy stochastic framework. The modeling strategy primarily relies on the intertemporal budget constraint, and this section outlines the main features of a basic model that fulfills the task. The rest of the model would fit in the standard international real business cycle literature, as in Gali and Monacelli (2008) for the case of a small open economy, or the recent literature on endogenous international portfolio allocation, such as Coeurdacier, Kollmann, and Martin (2010), Devereux and Sutherland (2010), and Tille and van Wincoop (2010).

This Section illustrates the new decomposition of the external budget constraint which will justify the empirical strategy of the paper; Appendix A and a *Supplementary Material* document available online provide the details of the derivations in this section and of the full theoretical model used for the results in Section 4.3. Since the empirical analysis of the paper hinges on the US, it is assumed that the world economy is formed by two symmetric countries: the domestic country, the US in the data, and the rest of the world.[3] The focus on economies that are able to affect the world equilibrium return rates and the stochastic discount factor implies that the conclusions of this research should be applied to the US and similarly large economies only. This is an important caveat to bear in mind when discussing the results, both theoretical and empirical, of the paper. A small open economy would face returns determined on the international markets instead, and its external debt dynamic must rely more on the traditional trade channel rather than the stochastic discount factor or returns movements analyzed in this paper.

2.1 Preliminary considerations

The only fundamental assumption necessary for the following theoretical results is the existence of a positive stochastic discount factor (SDF), $\Lambda_{t,t+1}$, that prices the available set of international securities; this is a fairly common assumption in macroeconomic and finance models. Given this assumption, the results of the paper would hold regardless of the degree of completeness of international financial markets.

The standard consolidated economy-wise budget constraint of a country is given by (1)

$$\underbrace{L_t - A_t}_{B_t} = R_t^L L_{t-1} - R_t^A A_{t-1} - \underbrace{(X_t - M_t)}_{D_t} \tag{1}$$

where B_t is the net foreign liabilities of the domestic country in terms of the domestic composite good, defined as the difference of foreign liabilities, L_t, and foreign assets, A_t; D_t is the trade balance defined by the difference between exports, X_t, and imports, M_t; R_t^L and R_t^A are the gross return rates on international liabilities and assets respectively. This equation closely resembles the standard definition of the current account, but it also takes into account the different returns on assets and liabilities and the valuation effects. In fact, the asset returns R_t^A include the valuation effects due to the exchange rate movements, while both R_t^A and R_t^L include also the standard capital gains due to changes in assets' prices.

The payoffs of the international securities can be the dividends paid by private equity or the returns on national government debt and private bonds. Given the framework of this paper, it is assumed that all these securities are bundled in an aggregate portfolio of domestic and foreign assets. This is done in order to have an easy mapping from the model into the data set provided by Gourinchas and Rey (2007a). In

[3] The only deviation from a perfectly symmetric model is different, but comparable, steady state output levels necessary to allow for a non-zero net debt position in steady state.

order to focus our attention on the sustainability of the overall external debt of a country, the possibility of domestically-run Ponzi schemes is ruled out. The remaining features of the model are of secondary importance for the characterization of the economy-wise budget constraint and the transversality condition. Different assumptions about the utility function of the consumer, the presence of labor and capital, or the role of the government would lead to a budget constraint for the domestic economy of the same form as in equation (1).

As illustrated by Gourinchas and Rey (2007a), equation (1) is a more correct representation of the American external constraint than the simple current account for two reasons. The first is that valuation effects on the gross positions have been shown to be quite large for the US. There is a crucial difference between the financial flows of net foreign liabilities and the actual change of the net asset position ΔB_t, and the accounting definition of the current account measures the financial flows rather than ΔB_t. The underlying statistical processes of the current account and of ΔB_t can sharply differ and their discrepancies are mostly driven by the valuation effects. For this reason, considerations about the required rebalancing of the current account should be based on the sustainability of the intertemporal budget constraint and not on the size of the current account. The second reason is that the rates of return on foreign assets and foreign liabilities might differ in a systematic way with Americans receiving a higher return on foreign assets than what they pay on liabilities. This is the exorbitant privilege of the US reported by Gourinchas and Rey (2007a). Curcuru, Dvorak, and Warnock (2008) dispute the actual size and importance of these return differentials, and they show how the measures of favorable excess returns for the US obtained in previous literature could be biased upwards due to the use of internally inconsistent revisions of the data on asset holdings and international asset flows. As it will be clear from the analysis below, debt sustainability per se does not strictly require positive excess returns and either case can be consistent with the theory presented in this paper. Furthermore, in steady state the return rates are equalized across assets by the optimality conditions of the representative agent; therefore, this feature of the model is more consistently justified by the evidence in Curcuru, Dvorak, and Warnock (2008).

2.2 Average returns and the intertemporal budget constraint

After applying the stochastic discount factor $\Lambda_{t,t+1}$, defined by the optimality conditions of the representative agent as the intertemporal ratio of marginal utilities between t and $t+1$, and the two optimal portfolio conditions $\mathbb{E}_t\left[\Lambda_{t,t+1}R^i_{t+1}\right] = 1$ for $i = A, L$, the budget constraint (1) can be expressed in intertemporal form as

$$B_t = \mathbb{E}_t \sum_{i=1}^{\infty} \Lambda_{t,t+i} D_{t+i} \tag{2}$$

where $\Lambda_{t,t+i} = \prod_{j=1}^{i} \Lambda_{t+j-1,t+j}$. The intertemporal budget constraint (2) also requires the transversality condition in (3) to hold

$$\lim_{T\to\infty} \mathbb{E}_t \Lambda_{t,t+T} B_{t+T} = 0 \tag{3}$$

Condition (3) closely resembles the standard transversality condition of a perfect foresight model, in which the discount factor would simply be the inverse of the risk-free interest rate, and it is an equilibrium implication of the optimal behavior of the representative agent of each country that does not allow foreign agents to run a Ponzi game.

Equation (2) defines the sustainability of the external debt position. Consistently with the intertemporal approach of the current account, current debt must be financed by positive expected future discounted trade surpluses. This equation justifies the common statement that some positive future trade surpluses are required to repay the current debt. As a matter of fact, recalling that $\mathbb{E}\Lambda B = cov(\Lambda, D) + \mathbb{E}\Lambda\mathbb{E}D$, positive trade surpluses are not even strictly necessary to obtain sustainability if the covariance between the SDF and trade surplus is positive and sufficiently large. The covariance term embeds the contribution of asset returns to sustainability; in particular, it captures the effects of returns due to the differences in their risk premia. Exploiting this fact and after a few other manipulations illustrated in Appendix A, the solution for B_t can be conveniently re-written as

$$B_t = \mathbb{E}_t \sum_{i=1}^{\infty} \frac{1}{R^F_{t,t+i}} \left[D_{t+i} - B_{t+i-1}\left(R^M_{t+i} - R^F_{t+i}\right) + B^M_{t+i-1} R^X_{t+i} \right] \tag{4}$$

where R^F_{t+1} is the risk-free rate between periods t and $t+1$, and $R^F_{t,t+i} = \prod_{j=1}^{i} R^F_{t+j}$; R^M_t is the average return rate on international securities $R^M_t = \frac{R^A_t + R^L_t}{2}$; R^X_t is the excess return $R^X_t = R^A_t - R^L_t$; and B^M_t is the average gross position in international securities $B^M_t = \frac{A_t + L_t}{2}$. The only assumption necessary to derive (4) is the existence of a positive SDF and of a risk-free rate in the economy that can represent the inverse of the expected value of the SDF.[4]

Equation (4) shows that besides the traditional trade channel and the valuation channel a la Gourinchas and Rey, represented by expected trade surpluses and excess returns respectively, there exist two other ways of financing higher external debt which have not received adequate attention in the literature so far. The first way, if the net debt position is positive, is a reduction of the expected spread between average and risk-free rates. This can be described as the *average return channel*. Then, a higher debt can also be matched by a

[4] This is implied by the portfolio conditions. Since R^F_{t+1} is risk-free and known at time t, it drops off the conditional expectations and we have

$$\mathbb{E}_t \Lambda_{t,t+1} = \frac{1}{R^F_{t+1}}$$

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lower expected risk-free rate, which implies a more favorable discounting of the other three components of the solution. The explicit use of the stochastic discount factor in the recursive solution of the constraint is crucial to highlight the new role of these two determinants of debt sustainability in addition to the first two channels usually included in the analysis of the rebalancing of the external constraint.[5]

Finally, this version of the solution of the intertemporal budget constraint presents an important advantage over (2) since it allows us to avoid the empirical difficulties related to the estimation of the SDF. In Section 4, I use a VAR model to explore some of the underlying mechanisms of the US external position sustainability by assessing how the factors in (4), especially excess and average returns, respond to shocks to the current debt. These responses will be sufficient to draw the desired conclusions about the role of these factors in the adjustment process of the American debt without explicitly requiring knowledge of the covariance between the SDF and trade surplus. Indeed, it is quite difficult to empirically measure this covariance term since the SDF is not uniquely identified if financial markets are incomplete and its characterization strictly depends on the specific choice of the utility functional form.

2.3 Linear approximation of the constraint

Some further insights about the solution of the external budget constraint can be obtained linearizing the model. Appendix A shows that two approaches to the first order linearization of (1) can be adopted. The linear approximation is taken around a non-zero steady state debt B_{ss} (in what follows, the steady state of a variable is indicated by the subscript ss). The portfolio equilibrium conditions imply that the steady state returns of the two assets are the same, hence $R^L_{ss} = R^A_{ss} = \beta^{-1}$, while the constraint implies that a positive trade balance must support the steady state debt, $X_{ss} - M_{ss} > 0$, if $L_{ss} - A_{ss} > 0$ and vice versa if debt is negative.

The first linearized version is related to the approach in Gourinchas and Rey and leads to the intertemporal budget constraint in (5)[6]

$$\hat{B}_t = \mathbb{E}_t \sum_{i=1}^{\infty} \beta^{i-1} \left(\hat{R}^*_{t+i} + \beta \hat{D}_{t+i} \right) \tag{5}$$

where $\hat{B}_t = L_{ss}\hat{L}_t - A_{ss}\hat{A}_t$ and $\hat{D}_t = X_{ss}\hat{X}_t - M_{ss}\hat{M}_t$ define the linear debt and trade surplus respectively and $\hat{R}^*_t$ is the sum of the linearized excess and average returns $\hat{R}^*_t = B^M_{ss}\hat{R}^X_t - B_{ss}\hat{R}^M_t$. Similarly to above, we define the linearized excess and average returns as $\hat{R}^X_t = \hat{R}^A_t - \hat{R}^L_t$ and $\hat{R}^M_t = \frac{\hat{R}^A_t + \hat{R}^L_t}{2}$. Equation (5) is

[5] It must be noted that in this representation of the solution the effects of the excess returns are also amplified by the size of the average gross holdings of international securities B^M_t.

[6] A derivation of the intertemporal constraint that more closely follows the approach in Gourinchas and Rey (2007b) is provided in the online Appendix. In particular in their original expression they take an approximation of the constraint around a deterministic trends which are replaced here by a constant steady state. Since the final results of the approximation exercise would be the same, only the time invariant steady state is considered for sake of simplicity. Appendix A describes in detail how equation (5) is obtained.

the linearized counterpart of (4).

In line with the approach proposed in this paper, the second version of the linearized intertemporal budget constraint is given by (6)

$$\hat{B}_t = \mathbb{E}_t \sum_{i=1}^{\infty} \beta^{i-1} \left(B_{ss} \hat{\Lambda}_{t+i-1,t+i} + \beta \hat{D}_{t+i} \right) \tag{6}$$

Gourinchas and Rey (2007b) find that a linearized measure of the external asset position of a country is a predictor of future excess returns and empirically demonstrate that a measure of trade and debt imbalances has a significant predictive power of the future excess returns on the American international portfolio. The intertemporal budget constraint in (2) has the theoretical implication that the current net foreign liabilities should predict average returns in addition to excess returns. I show next that $\hat{\Lambda}$ in (6) embeds these two components in the linearized model as well.

In general equilibrium, the return rates are endogenously tied to the stochastic discount factor by the optimal portfolio choices. As explained by Tille and van Wincoop (2010) and Devereux and Sutherland (2010), this implies that the expected excess returns are zero up to the first order of approximation leaving only the terms $B_{ss}\hat{R}^M_{t+i}$ in (5). Combining (5) and (6), we find that the present value of the average returns reflects the present value of the SDF

$$\mathbb{E}_t \sum_{i=1}^{\infty} \beta^{i-1} \hat{\Lambda}_{t+i-1,t+i} = -\mathbb{E}_t \sum_{i=1}^{\infty} \beta^{i-1} \hat{R}^M_{t+i} \tag{7}$$

An increase in net debt, $\hat{B}_t > 0$, that is not offset by a change in the future trade flows, $\mathbb{E}_t \sum_{i=1}^{\infty} \beta^i \hat{D}_{t+i} = 0$, must be offset by an increase in the SDF, $\mathbb{E}_t \sum_{i=1}^{\infty} \beta^{i-1} \hat{\Lambda}_{t+i-1,t+i} > 0$, which raises the net present value of the stream of steady state trade surpluses. Equivalently, in a country with a positive net debt in steady state $B_{ss} > 0$, equation (7) implies that future average returns must fall, thus lowering the interest payments for a given value of debt and allowing the steady state trade surpluses to cover the interest bill of a larger debt. The average returns have a larger role in the external rebalancing than the excess returns as it is reflected by the SDF. However, equation (7) also implies that the average return spreads in (4) must be zero in expectations up to a first order approximation as it is for the expected excess returns; both excess return and average return channels are, therefore, confined to secondary effects only.

This result is a re-interpretation of the conclusions of Tille and van Wincoop (2010) and Devereux and Sutherland (2010), which state that Gourinchas and Rey's empirical evidence of predictability of the US excess returns captures only higher order components of the returns. Even though the expected excess returns are zero up to the first order of approximation, some predictable effects can be found in the subsequent higher order terms.[7] The response functions to the debt shocks in the VAR models analyzed in Section 4 point out that the effects of both types of returns in the Gourinchas and Rey sample become negligible when data is not detrended, even though the predictability of excess and average returns is found with detrended data. This result is consistent with the theoretical explanation of this Section and the same conclusions are also confirmed when the VAR is estimated using simulated data obtained from the model.

3 Sources and Description of the Data

In the empirical analysis of the next section, I use quarterly data on trade from the National Income and Product Accounts (NIPA) tables, and the data from Gourinchas and Rey (2007b) for the financial series.

The NIPA tables published online by the Bureau of Economic Analysis (BEA) provide the nominal series of imports, exports, international income payments and receipts, GDP, and consumption. The consumer price index series from the online database of the Federal Reserve Bank of St. Louis, the Federal Reserve Economic Data (FRED), is used to construct the inflation series. The three-month treasury bill rate is used as a proxy risk-free interest rate. Gourinchas and Rey (2007b) provide the series of the US nominal foreign assets and liabilities holdings and household net worth; they also provide the nominal rates of return for the two categories of international assets.[8] This data allows us to construct the ratio to net worth and the ratio to GDP series that are used in the VAR regressions.

The common sample covered by both sources dates back to the beginning of the 50s until 2004, precisely from 1952:3 to 2004:1. The samples of the VAR regressions are then limited by the real exchange rate series, which can be obtained only since the 70s. I use the series constructed by Bianchi and Civelli (2014), which is very similar to the major currencies real exchange rate series reported by FRED, but has the advantage of starting from 1970:1. Whenever necessary, the series are seasonally adjusted.

The timing of the observation of the data needs a further clarification, in particular because it is important for the following empirical applications. Gourinchas and Rey provide the end of period nominal values of the portfolio allocations, computing the flows of assets and the valuation rates over the period (ΔF_{t+1} and χ_t, respectively, in equation 8). In the notation of the equation, B_{t+1} refers to the end of period $t+1$ value of net assets, while ΔF_{t+1} are the flows accrued during period $t+1$ (that is from the end of period t, when

[7] In particular, possibly non-zero constant expected excess returns proportional to the variance of the innovations of the model correspond to the second order of approximation; while time-varying predictable changes in $\mathbb{E}_t R^X_{t+1}$ correspond to the third-order components, which in turn are a function of the first order component of the state variables of the model. I derive the second order approximation of the constraint which includes these effects too in the online *Supplementary Material*.

[8] The empirical analysis of this paper has been made possible by the availability of Gourinchas and Rey's data set, which is found on the authors' webpages. The reader is invited to make reference to their paper for a detailed description of the data.

B_t has been observed, to the end of period $t+1$).

$$B_{t+1} = B_t + \Delta F_{t+1} + \chi_{t+1} B_t \tag{8}$$

The same timing of ΔF_{t+1} must be assumed for the trade balance and income payments and receipts from the NIPA data since ΔF_{t+1} conceptually corresponds to the current account. Consistently, the real exchange rate, S_t, the consumer price index, the prices of assets and liabilities, as well as the growth rate of household net worth used in the next section are observed at the end of each quarter.

The limitation in the sample suitable for the VAR regressions due to the real exchange rate series is relatively unimportant. In some sense, it helps to mitigate possible structural breaks in data that may have followed the increase in openness of the world economy after the collapse of the Bretton Woods system. However, the results I obtain excluding the real exchange rate from the basic specification of the VAR are not particularly affected by differences in the sample sizes.

3.1 The Debt to Net Worth Ratios

For the empirical analysis of the paper, I consider the debt to net worth ratio, instead of the real debt obtained by simply dividing the nominal debt by the price level. This is done because it is common practice in literature to address questions about debt dynamics using standardized values of debt; I adopt the same standardization used by Gourinchas and Rey (2007b). A similar standardization is used also by Bohn (1998) who takes the ratio of debt to GDP. The results in this and the next section are identical whether the variables are divided by net worth or GDP.

Consistently with the notation above, let lower cases indicate the variables in ratio format. The budget constraint corresponding to (1) would be

$$\underbrace{l_t - a_t}_{b_t} = R_t^l l_{t-1} - R_t^a a_{t-1} - \underbrace{(x_t - m_t)}_{d_t} \tag{9}$$

where the return rates R_t^a and R_t^l are real returns that incorporate the growth rate of household net worth. It is worth stressing again that the asset returns depend on the depreciation rate of the real exchange rate too, denoted γ_t henceforth.[9] It must be noted that considering the ratios to net worth of the variables in the

[9] It is easy to link these return rates to the real returns in Section 2. Denoting g_t the real growth rate of net worth, we can express R_t^a and R_t^l as

$$\begin{aligned} R_t^a &= \frac{R_t^A}{(1+g_t)} = \frac{(1+\gamma_t)}{(1+g_t)} R_t^{*A} \\ R_t^l &= \frac{R_t^L}{(1+g_t)} \end{aligned}$$

variable	*alpha*	specification	p	stat	c.v.	sample
a_t	y	α and δ	14	-1.45	-3.43	52:3-04:1
l_t	y	α and δ	8	0.87	-3.43	52:3-04:1
d_t	n	α	0	-0.58	-2.89	52:3-04:1
S_t	n	α	3	-2.99	-2.89	70:1-04:1
bb_t	y	α and δ	0	-0.78	-3.43	52:3-04:1
bb_t	y	α	0	0.90	-1.65	52:3-04:1
b_t	y	α and δ	18	-0.63	-3.43	52:3-04:1
b_t	y	α	18	1.32	-1.65	52:3-04:1

Table 1: Augmented Dickey-Fuller Tests. The column *alpha* indicates a constant is assumed in the true generating process of the variable. The column *specification* indicates whether a constant (α) and a trend (δ) are included in the test equation. p are the lags of the augmented component. *Stat* is the value of the test statistic, while *c.v.* is the corresponding critical value at 5% level. The variables tested are the real exchange rate S_t and the ratio to household net worth of assets a_t, liabilities l_t, trade balance d_t, debt inclusive of interest payments bb_t, and net of interest payments b_t

budget constraint does not invalidate any of the results discussed in the theoretical section of the paper.[10]

Before turning to the VAR analysis, I check for the stationarity of the series included in the budget constraint, which also appear in the transversality condition and the VAR. The augmented Dickey-Fuller test for unit roots is used; Table 1 reports the results of the tests.[11]

The top panel of Table 1 illustrates the results for assets and liabilities, trade balance and real exchange rate, S_t. The bottom panel illustrates the results for the debt inclusive of interest payments, $bb_{t-1} = R_t^l l_{t-1} - R_t^a a_{t-1}$, and net of interest payments, b_t, under alternative specifications of the test. All the series, with the exception of the real exchange rate which has a marginally significant statistic, seem to be non-stationary, and the unit root tests are not rejected with large margins. The same conclusions are confirmed for almost all the other specifications of the Dickey-Fuller test.

The tests on the debt series aim to provide some evidence on its stationarity rather than a proof of

where it should be clear from the notation adopted so far that R_t^{*A} is the real return rate of the foreign assets in terms of foreign goods.

[10] Restating the budget constraint in these terms would simply determine an adjustment in the definition of the discount factor and steady state values of the return rates. For instance, the intertemporal constraint would become

$$b_t = \mathbb{E}_t \sum_{i=1}^{\infty} \lambda_{t,t+i} d_{t+i}$$

where λ is the discount factor corrected to account for the growth rate of net worth g_t.

[11] Hamilton (1994) describes the procedure to follow in order to choose the specification of the test equation and provides the correct tabulation of the values of the t-test statistic in each case. The test is based on a regression which, in its most general form, has the following specification:

$$\Delta y_t = \lambda_1 \Delta y_{t-1} + \lambda_2 \Delta y_{t-2} + ... + \lambda_p \Delta y_{t-p} + \alpha + \rho y_{t-1} + \delta t + \varepsilon_t$$

I select p according to the following procedure: From the auto-correlogram of Δy_t, one can single out the highest value of p such that the $(p+1)th$ term in the test equation is presumably not significant, this p should correspond to the lag after which the auto-correlogram fades out. Given this, the joint significance of the $(p)th$ and the $(p-1)th$ terms in the regression is tested: If the F-test does not reject the significance of the two terms at five percent of confidence, the procedure stops here and Δy_{t-p} is included in the test regression. If, on the other hand, the test rejects it, Δy_{t-p} is discarded and the same steps are repeated for the next lag.

the existence of the unit root per se. Estimating the VAR with Bayesian techniques in the next section overcomes the issues related to the presence of unit roots in the variables of the model, at least for the correct computation of the impulse response functions and their bands. However, since in the VAR literature it is a common first step to check for the integration of the time series included in the regression, and given that the random walk is the baseline case of non-stationarity, it is worthwhile to run these tests.

Showing that the debt series is integrated is a simple way to prove that the debt is sustainable because it would conveniently satisfy the transversality condition too. This was the typical approach taken in the earlier literature on sustainability. Here, I am not directly focusing on sustainability, but rather on the relative comparison between the channels of sustainability. Furthermore, the concept of sustainability is formally defined by the comparison of the growth rate of net debt and the contraction rate of the discount factor. Since explosive debt accumulation rates (i.e. rates larger than one) are perfectly plausible, it seems preferable not to impose any particular structure, least of all a unit root, to the long-run dynamics of debt.[12]

4 VAR Analysis

The systematic contribution of the excess returns to the rebalancing mechanism of the debt accumulation documented by Gourinchas and Rey (2007b) for the US has an intriguing interpretation. If trade surpluses are systematically compensated by favorable excess returns, then the burden of the solvency of the external debt of a country does not have to exclusively rely on the trade balance. This self-adjusting mechanism can be triggered, for example, by the depreciation of the real exchange rate that causes positive excess returns while preparing the way to the long-run positive surpluses. However, the theoretical considerations of Section 2.1 suggests that other channels should be more important than the excess returns for the adjustment of the US external position even in the Gourinchas and Rey sample period. Moreover, excess returns as well as average return channel must be secondary relative to the trade channel and the stochastic discount factor effects. This section aims to reconcile the empirical evidence of Gourinchas and Rey with these other theoretical results. The main result is that detrending the data magnifies the importance of the excess and average return channels in a way that we could define as spurious with respect to the formal definition of the intertemporal constraint in (4).

The empirical strategy adopted is based on the comparison of the responses of the four determinants of the external constraint to a shock to the current debt for two versions of the same VAR model. The first

[12] I return to this point in the next section. This paper helps in stressing the point that the cointegration analysis is probably not the most convenient approach to model the relevant long-run relationships among the variables in this VAR framework when the goal is to discuss the sustainability of debt.

For a very comprehensive cointegration study of the adjustment dynamics of the U.S. current account, see for example Holmes and Panagiotidis (2009).

version is estimated using non-detrended data, and it can be thought of as the empirical counterpart of the non-linear model in (4); I refer to it as the *non-detrended model* in Section 4.1. In the second version, trends are excluded from the analysis as in Gourinchas and Rey; this is the *detrended model in* Section 4.2, and it can be seen as the counterpart of the linearized model in (5). The specification of the VAR will include six variables in ratio to the households' net wealth: trade balance, net foreign debt, exchange rate depreciation, excess returns, the average return spread, and the risk-free interest rate.

A clear difference is found in the responses of the two models. While the response of excess returns is significantly positive in the detrended model, it is very small and not significant in the non-detrended one. The response of the average return spread to a debt shock is zero in both models, reflecting the prediction that the average returns mirror the risk-free rate. On the contrary, the risk-free rate, which represents the inverse of the expected discount factor in the VAR, exhibits negative and significant responses in both cases. The conclusion of this analysis is that the majority of the adjustment process of the US external position is driven by movements in the discount factor rather than the other channels.

Finally, Section 4.3 explores the validity of these conclusions beyond the American dataset using simulated data from a general equilibrium open-economy model in the VAR analysis. This final exercise makes two main points. First, the evidence on the importance of the average returns and risk-free rate in the adjustment process is not specific to the US sample and can be generalized to a broad set of cases. Second, spurious predictability of the excess returns is obtained with simulated data in presence of sufficiently persistent deviations from the steady state of the model implied by the balanced growth path assumption. As discussed below, the filtering of the dataset to remove the trends introduces distortions in the relations between the variables of the model conceptually equivalent to a miss-identification of the correct balanced growth path steady state.

4.1 Non-Detrended Model

I consider first the non-detrended model. The goal of this exercise is to show that neither future excess returns nor average return spreads are predicted by the debt shocks in the Gourinchas and Rey sample, while these shocks induce a response of the expected stochastic discount factor consistent with the rebalancing of the intertemporal budget constraint (4).

The main specification I study includes six variables: the net foreign debt position b_t, the trade surplus d_t, the excess returns R_t^X, the depreciation rate of the real exchange rate γ_t, the three-month treasury bill rate R_t^F, and the spread between average returns and risk-free rate $R_t^M - R_t^F$. The three-month US treasury bill rate is used as a proxy of the actual risk-free rate of the economy in order to capture the effects of the SDF on

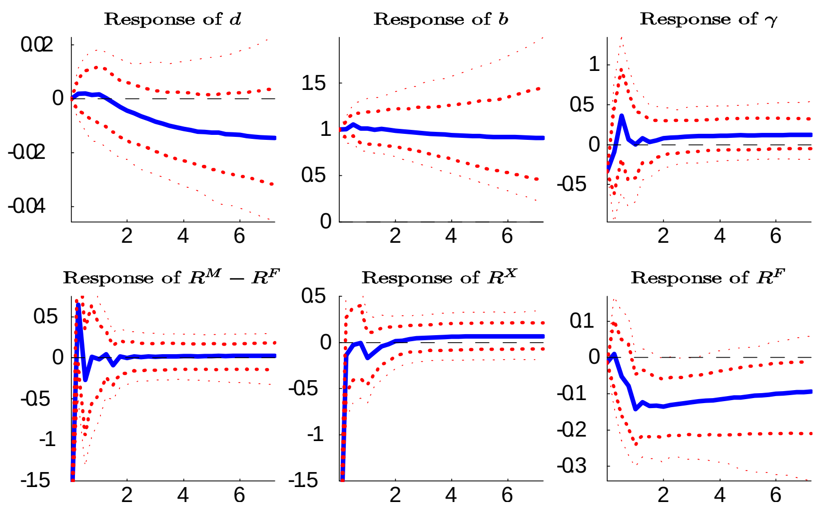

Figure 1: Response functions to a b unit shock. Non-detrended VAR model - baseline specification. Band intervals correspond to the 14/86th and 5/95th percentiles. Years from the impulse on the x-axis. The variables in the model are the net foreign debt position b_t, the trade surplus d_t, the spread between average returns and three-month treasury bill rate $R^M - R^F$, the excess returns R_t^X, the depreciation rate of the real exchange rate γ_t, the treasury bill rate R_t^F. The limits of the vertical axes for the $\left(R^M - R^F\right)$ and R^X responses are set to (-1.5). The ordering of the variables in the Cholesky decomposition is $\left(d,\ b,\ \gamma,\ R^M - R^F,\ R^X,\ R^F\right)$.

the intertemporal constraint.[13] In the following set of results, I estimate the VAR with Bayesian techniques using four lags and a Minnesota-type prior over the Gourinchas and Rey sample period 1970:1-2004:1, which guarantees direct comparability with their results.[14] The structural shocks for the impulse response functions are identified by a recursive Cholesky scheme based on the ordering $\left(d,\ b,\ \gamma,\ R^M - R^F,\ R^X,\ R^F\right)$. This identification scheme is justified by two main reasons. First, the real variables block is separated from the financial block; the only real variable of the model is the trade surplus, and it is assumed to respond to the financial variables with one lag. For example, a similar assumption is common in the monetary VAR literature where the policy rate is ordered after inflation and output gap. Second, within the financial variables, debt is placed first so that the three return components are allowed to respond on impact to a debt shock. This assumption is in line with the first order characterization of R^X and $R^M - R^F$ as *i.i.d.* variables. There is little guidance for the relative position of these three components, but fortunately the impulse response functions are very robust to different specifications of this subset of variables.

Figure 1 illustrates the full set of response functions for the model. The debt shock triggers a response of R^F consistent with a positive contribution of the SDF to the adjustment process of debt. The risk-free rate

[13] As mentioned in Section 3.1, variables in ratio to net worth are used in the empirical analysis with the exception of the return rates, used in real terms.

[14] The parameter of the tightness for the own lags of a variable, henceforth ϕ_0, is set to .4. The relative tightness parameter for the other variables' lags, henceforth ϕ_1, is set to .5. The lags have an harmonic decay with parameter 2, and the tightness parameter of the exogenous variables (only the constant here) is set to 10^3. The prior mean of the first lag is 1 for every variable. I leave a detailed discussion of the choice of the prior parameters and of the sensitivity analysis of the results for the online Appendix.

decreases, and the response is statistically significant (the two pairs of dotted lines correspond to the $14/86th$ and $5/95th$ percentile intervals of the posterior distribution of the response functions computed by Monte Carlo integration). The response of b is quite persistent but clearly mean-reverting. At impact, the excess returns display a very large drop, but then R^X promptly reverts toward zero and remains non-significant after the second quarter.[15] Similarly, the response of the spread is large on impact and for the next couple of periods, but it is not significant again after that. The median response of the trade balance is mostly negative with the exception of the initial two or three quarters in which half of its posterior distribution lies in the positive region. The intertemporal approach to the current account expressed by the budget constraint (2) relies mostly on the response of future trade surpluses to an increase in the current net debt position. Even though the response of d is only partially in line with the intertemporal approach, the initial portion of the response of d is satisfactory, and also in the medium/long horizon there is a non-trivial positive probability of having some future positive trade surpluses, and this would be enough to guarantee the existence of states of the world in which the trade balance turns positive. Furthermore, switching the relative ordering of b and d in the identification scheme of the Cholesky decomposition further increases the portion of positive response.[16]

The main determinant of the adjustment mechanism is found to be the movements of the (expected) stochastic discount factor, represented by the risk-free rate responses. The contribution of the excess returns is positive, but not significant, and consistent with the characterization of second order or higher effects of the theories presented by Tille and van Wincoop and Devereux and Sutherland. That this evidence is compatible with these theoretical results can also be inferred from the difference between the large jump on impact, ascribable to the theoretical *i.i.d.* first order component of the excess returns, and the smaller magnitude of the subsequent part of the response. The analysis of the detrended model in the next section reinforces these conclusions. On the contrary, the average return spread, which also should have only secondary effects, fails in displaying a clear contribution to the rebalancing of the external debt even after including the discount factor in the model. As suggested by the theory in Section 2.3, the predictability of the average returns is mostly related to predictable movements of the discount factor/risk-free rate.

Two more pieces of evidence deserve a further comment. The first is about the persistent median trade

[15] The impact response is around -3. In the figure, the lower limit of the vertical axis is set to -1.5 in order to provide a better illustration of the response in the following periods too. A similar truncation of the y-axis is applied for the responses $R^M - R^F$.

[16] Several robustness checks to the baseline specification are reported in the online *Supplementary Material*. In particular, I consider three other specification. First, a specification without the risk-free rate to isolate the effects of excess returns and spread alone; their responses revert quickly to the zero line and remain there forever. Second, consumption is added to the baseline in order to improve the approximation of the expected discount factor in the model; the response of debt gets definitely less persistent, while the other responses remain substantially unchanged. Third, the spread is replaced with R^M in order to check whether the average returns mirrors the response of the risk-free rate as predicted by the theory for its first order component; the zero response of the spread is explained by the similarity in the responses of these two variables.

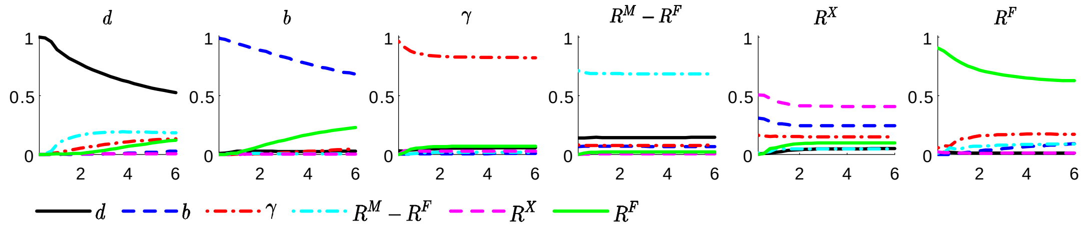

Figure 2: Variance decomposition for the non-detrended model. Horizon of the forecast in years on the x-axis. Identification scheme of the shocks based on ordering $\left(d,\ b,\ \gamma,\ R^M - R^F,\ R^X,\ R^F\right)$.

deficits in Figure 1, which may cast some doubt on the actual contribution of the trade channel to the adjustment of the US debt. It is interesting to explore whether those median deficits are the outcome of the rapid acceleration in debt and the prolonged current account deficit of the last part of our sample. In order to evaluate this possibility, I re-estimate the VAR over the sample 1970:1-1990:1, in which debt and trade deficit were not much of a concern yet. The median trade balance response is now positive at each horizon, and the reversion of debt to the mean after the shock is faster.[17] The second piece of evidence is the variance decomposition of the model presented in Figure 2. Two observations are interesting about this figure. First, the risk-free rate shocks, but not the excess return nor the average return spread shocks, explain a large share of the debt variance in the longer term. Second, about 30% of the variability of R^X is actually attributed to the debt shocks at every horizon, and a good portion is then explained by the exchange rate shocks. Both the observations are in line with the impulse response analysis and the theoretical predictions discussed above.

Before moving to the detrended model, a final clarification about the relation between sustainability and persistence of the debt shocks is required. While mean-reverting shocks are associated to sustainability by definition, permanent or mildly explosive shocks should not be automatically associated to unsustainable debt. Obviously, highly persistent shocks make the solvency constraint more binding, but unsustainability corresponds to explosive trajectories that violate the transversality condition (3).[18] For this reason, interpreting the evidence of Section 4 as suggesting that excess returns are predictable when the debt is sustainable and unpredictable when sustainability is compromised, with the resulting implication that excess returns play a primary role in the sustainability mechanism, would not be correct.

[17] The responses of the other variables are very similar to the baseline model. The results are reported in the online Appendix.
[18] See Bohn (1995) and his related work.

4.2 Detrended Model

This section studies the detrended version of the VAR model. Following the approach in Gourinchas and Rey (2007b), the variables are constructed by taking the log-deviations from a trend computed applying an Hodrick-Prescott (HP) filter to the series of the model.[19] Since it is important to show that the VAR framework adopted in this analysis is adequate to make a comparison with the Gourinchas and Rey approach, I use this framework to obtain a result on the predictability of the excess returns explicitly comparable to their type of evidence. I then extend the predictability result to the average returns as well, as predicted by (5), corroborating the new theoretical implications of (7) (i.e. the matching of stochastic discount factor and average returns). The interest in the average return suggests focusing on a specification including the following six variables $\left(\hat{d},\ \hat{b},\ \gamma,\ \hat{R}^m,\ \hat{R}^x,\ \hat{R}^F\right)$. Alternative specifications would lead to exactly the same conclusions. The prior parameters and the identification strategy are the same as in the non-detrended case.

It is not obvious whether de-trending is the most appropriate strategy to deal with the long-run properties of the variables in the VAR. The most common alternative would be to use a Vector Error Correction Model (VECM), which exploits cointegrated relations between integrated series. Although admissible in this context, there is an important consideration that prevents a standard use of the cointegration techniques. Assets, liabilities, net debt, as well as imports and exports are clearly non-stationary in the data. However, these series could plausibly grow at an exponential rate instead of being integrated of order one, and treating them as $I(1)$ processes would not be any more correct than de-trending them. In addition, this point is supported from a theoretical perspective by the transversality condition in (3). Not only are exponential growth rates of debt contemplated in this condition, but they also represent the most relevant case in terms of sustainability. Since the long-run analysis of the non-detrended model remains valid regardless of the assumptions made about the data generating process of debt, it is more germane to adopt the Gourinchas and Rey's approach given the goal of comparing my analysis to theirs.

Figure 3 illustrates the responses of the six variables of the detrended model to a unit debt shock. The shock is mean-reverting, and it is followed by a significant depreciation of the exchange rate and positive, significant excess returns up to four years after the shock. The average return is positive for about one year, but then turns significantly negative over the medium to long horizon as expected based on the external constraint. The introduction of the risk-free rate is not necessary to generate these responses and the same results are found under more parsimonious specifications. The risk-free rate falls for a prolonged period which

[19] They choose a smoothing parameter value of 2400654, which sets a gain of .7 at the frequency of 200 quarters. This is a very high smoothing parameter of the HP filter, which eliminates only the very slow-moving components of the series. Since the returns are stationary, the estimated trend for such a high parameter corresponds to their sample average. The exchange rate depreciation rate γ is computed as the quarterly log-difference of the real exchange rate index and it is the same used in the non-detrended model.

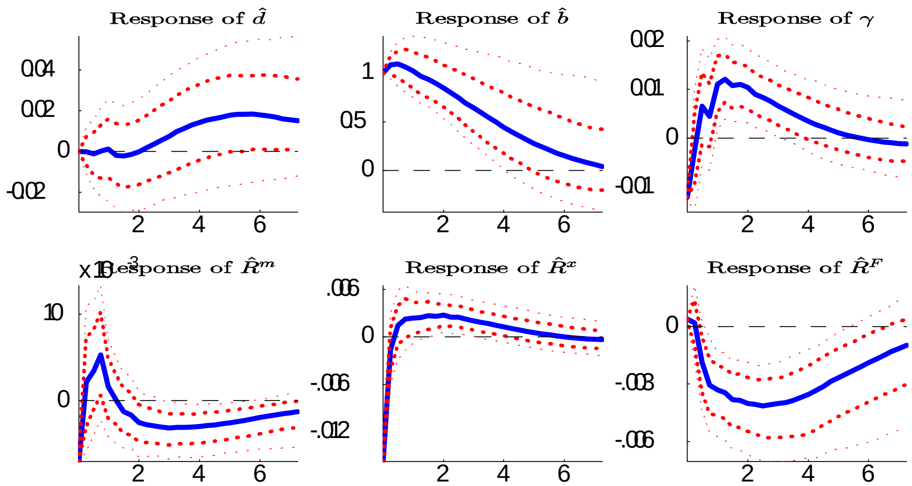

Figure 3: Response functions to a $\hat{b}$ unit shock. Detrended model. The variables in this specification are the log-deviations of net foreign debt $\hat{b}_t$, trade surplus $\hat{d}_t$, depreciation rate of the real exchange rate γ_t, average return rate $\hat{R}_t^m$, excess returns $\hat{R}_t^x$, and of the real risk free rate $\hat{R}_t^F$. The identification ordering is the same as for the non-detrended model. Band intervals correspond to the 14/86th and 5/95th percentiles. Years from the impulse on the x-axis.

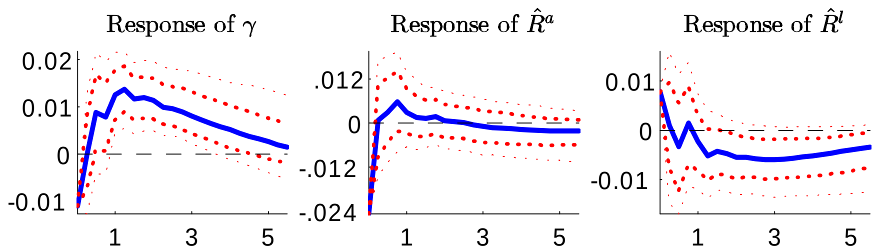

Figure 4: Responses of γ, $\hat{R}^a$, and $\hat{R}^l$ to a $\hat{b}$ unit shock. Main specification of the detrended VAR model, in which $\hat{R}^a$ and $\hat{R}^l$ replace $\hat{R}^m$ and $\hat{R}^x$. Years from the impulse on the x-axis.

means that the SDF is expected to increase; the median of the trade balance response is positive, with the exception of a few initial quarters, and a large portion of the posterior distribution of this response is in the positive region too. Overall these responses describe a situation of predictability of the excess and average returns consistent with the results of Gourinchas and Rey. The joint evidence from the non-detrended and detrended models demonstrates that detrending the data causes a spurious overestimate of the importance of the excess return channel relative to the effects of the SDF. The effects of excess and average returns are of secondary importance because they are smaller in size, but also because they are less effective in activating the adjustment process of debt.

The relationship between the real exchange rate depreciation and $\hat{R}^X$ deserves some further attention. The valuation effects are typically attributed to movements of the exchange rate and the foreign asset returns, and this is the interpretation given by Gourinchas and Rey too. I study this link more specifically estimating a VAR model in which excess and average returns are replaced by the asset and liability returns in order to evaluate their relative contribution to the two return channels. Figure 4 illustrates the separate response functions of $\hat{R}^a$ and $\hat{R}^l$ to a $\hat{b}$ shock. At impact, the real exchange rate appreciates followed by a long lasting significant depreciation, but $\hat{R}^a$ follows it only for the first six to eight quarters; in the medium and longer term, $\hat{R}^a$ becomes slightly negative even though not particularly significant. On the other hand, the liability return rate $\hat{R}^l$ turns negative one year after the shock, and it basically drives the positive excess returns and the negative average returns for the horizon between one and four years in Figure 3. The response of the excess returns in the medium term are then mostly attributable to the response of $\hat{R}^l$; although it can be quite large, this effect cannot be directly attributed to the exchange rate valuation channel per se.

In comparison to the decomposition of the excess returns reported by Gourinchas and Rey, in which asset returns explained more than 70% of the excess returns, my analysis points out a more critical role of the liability returns. This is an important distinction that puts the valuation channel in a different perspective. Valuation effects related to the exchange rate are important determinants of the composition of the excess returns, because an exchange rate depreciation has a direct effect on $\hat{R}^a$, but other less direct transmission channels can be at work too. One of these forces can be found in a saving-glut-type argument for instance: a shift of the world demand for international securities toward the American liabilities would cause an increase in the American debt and, at the same time, a decrease in the return rate of the US liabilities. Part of the depreciation of the exchange rate not directly linked to the excess returns would follow due to the activation of the traditional trade channel. A shock to the international debt of a country or to its trade surplus basically affects the distribution of wealth across countries. Changes in relative wealth influence the general equilibrium outcomes of the global economy and this has inevitable repercussions on assets prices that are only partially captured by the exchange rate.

4.3 Simulated-Data Model

The evidence found in the previous two sections could be specific to the US sample and time period covered by the Gourinchas and Rey dataset. In this last exercise, I assess the broader validity of these results by repeating the VAR analysis on simulated data obtained from a small-scale open-economy stochastic general equilibrium (DSGE) model. Although rather stylized, this theoretical framework provides already some clue to explain the underlying mechanism of the results, reinforcing the conclusions of the paper in two ways. First, detrending introduces a distortion in the relations between the variables of the VAR model that causes spurious predictability of the excess returns. These distortions occur in presence of sufficiently persistent deviations of the simulated data from the steady state of the theoretical model defined by the balanced growth

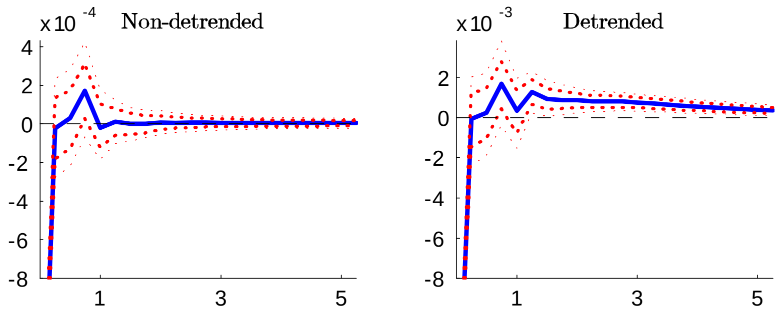

Figure 5: Response functions of the excess returns to a unit debt shock. VAR analysis with simulated data for the two data treatments: Non-detrended model (on the left) and Detrended model (on the right). The two specifications of the VAR model correspond to those in Figure 1 and Figure 3. The identification ordering is the same as in the previous VAR models. The vertical axes are truncated to improve the visualization of the effects after the large impact response. Band intervals correspond to the 14/86th and 5/95th percentiles. Years from the impulse on the x-axis, defined as 4 periods in the simulation.

path (BGP) assumption, which guarantees an appropriate use of linear approximation solution techniques. The size of these effects are determined by the relative size and prevalence of the deviations from the steady state in the observed sample. Second, the average return rate matters and reflects the movements of the SDF; this conclusion is very robust to different calibrations.[20]

The model used to generate the simulated data is a standard two-country two-good two-asset DSGE model very similar to that in Tille and van Wincoop (2010). Data is generated for a large open-economy domestic country with access to two internationally traded securities, corresponding to A and L in (1). Financial markets are assumed to be incomplete, since the model includes more shocks than assets. The model is calibrated to broadly reflect the volatility of the US series included in the VAR analysis; it is solved up to the first order accuracy around a steady state in which the net foreign debt is positive, and then simulated for a sequence of shocks. For sake of brevity, the details about model, calibration, and simulation are left for the online *Supplementary Material*.[21]

I focus on a particularly illustrative example in which persistent deviations from the steady state are introduced in the simulation by mechanically imposing a sequence of negative shocks to the domestic subjective discount parameter.[22] This case is designed to replicate the observed shift in the US debt position, which

[20] I would like to thank an anonymous referee for suggesting the use of simulated data to confirm the validity of the empirical analysis.

[21] The two countries are symmetric in structure, except for the steady state levels of output which are not necessarily identical. This condition is necessary to allow for a non-zero net foreign debt in steady state. The sizes of the two countries would be comparable though, which implies a model suitable for large open economies. Each economy receives an endowment of a country-specific good in each period. The two goods are freely tradable, but not perfect substitutes, and the countries exhibit a home bias in consumption that allows for meaningful dynamics of the real exchange rate. The stationarity of the steady state distribution of the net debt position is regulated by an Uzawa-type mechanism.

[22] The model is simulated for a long series of innovations of $10,000$ periods drawn form a joint normal distribution. Half way through the simulation, the domestic preferences shock is set to $-.01$ and it assumed to be linearly re-absorbed for the following 500 periods. A sample of 1000 periods centered around this block of persistent shocks is then used for the VAR results illustrated in Figure 5.

can be considered quite large and persistent with respect to the structure of the model, allowing the system to smoothly revert after that towards a more regular regime. In this way, the analysis clearly highlights the differences between detrended and non-detrended models; nevertheless, reproducing the spurious effects of detrending would be fairly common with any sequence of random innovations. The effects would typically be smaller, but still significant, as documented in the online Appendix.

Figure 5 compares the responses to a unit shock to the foreign debt of the excess returns for the two data treatments; the responses of the other return rates are reported in the online *Supplementary Material*. After the large drop on impact, the excess returns response is not significant in the non-detrended model; on the contrary, the median response in the detrended model turns consistently positive following the initial drop, and it is highly significant after one year (corresponding to 4 simulated periods). Similarly, the simulated data analysis corroborates the empirical VARs results for the other return components. The response of the risk-free rate is negative and largely significant in both the specifications of the model. The response of the average return spread is small and not significant. Finally, the response of the average returns is significantly negative as expected from the theory, even though it falls short in persistence with respect to the risk-free rate.

This exercise provides an effective explanation of the spurious predictability of the excess returns discussed in this paper. The spurious effects are the by-product of applying a filter to the data to remove in-sample trends. The filter interprets persistent but plausible deviations of normalized variables from their balanced growth path as slow-moving time components, which it smooths out altering the relations between the variables of the model. Conceptually, this is empirically analogous to a miss-identification of the correct balanced growth path steady state of the model.

5 Conclusions

Following Gourinchas and Rey (2007b), literature has mostly focused on the role of trade and valuation channels in the adjustment mechanism of the external position of a country. The main theoretical contribution of the paper is to show that a current debt position can be offset by two additional factors: the average return spread and movements in the stochastic discount factor, which has not received the due consideration by this strand of the literature. The decomposition applies to the budget constraint with no need of approximation. However, a first order linearization shows that both excess returns and average return spread only have second order effects on the debt dynamics, leaving most of the burden of adjustment to the trade channel and the SDF. The linearization also illustrates that average returns perfectly mimic the SDF up to a first order approximation. Second order predictable effects of the excess returns were theoretically described already by Tille and van Wincoop (2010) and Devereux and Sutherland (2010), this paper extends the same analysis and conclusions to the average return spread too.

On the empirical ground, the paper focuses on the sample of US data used by Gourinchas and Rey and it illustrates that, by detrending the data, their approach incorrectly overestimates the role of the excess returns. Using a VAR framework to contrast detrended with non-detrended data, I assess the relative contribution of the adjustment channels for the US in this sample period. Three main results are found: First, the adjustment of the external position is mostly supported by the SDF. Second, the contribution of the average return spread is not significant because the average returns reflects the SDF. Third, the paper points out the risk of spurious results in favor of the excess return channel in the detrended analysis and it reconciles the large predictability of excess returns found by Gourinchas and Rey (2007b) for the US with their theoretical characterization as second order effects. Finally, I show that these conclusions are not limited to the US data repeating the VAR analysis with simulated data from a standard two-country DSGE model. The simulation cautions us about the negative consequences of detrending data when the deviations of data from the BGP are persistent.

In conclusion, two important caveats must be mentioned. First, the effects of the stochastic discount factor and of the average returns are plausible only when an economy is big enough to affect asset returns internationally. For small open economies, we could expect these new channels to be less effective and the traditional trade channel to matter the most. Second, even though the theoretical decomposition of the paper allows us to study the factors affecting the rebalancing mechanism of the external constraint of a country, very little can be said about the actual sustainability of the trajectory of the US foreign debt from this type of analysis. Despite solvency being perfectly defined in theoretical terms by the intertemporal budget constraint (2) and the transversality condition (3), a full assessment of sustainability requires some sort of empirical knowledge of the SDF, which is unobservable and not even uniquely defined if markets are incomplete. This is a challenging question left for future investigation.

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APPENDICES

A Derivation of the Theoretical Results in Section 2

This Appendix provides a more exhaustive derivation of some of the results in section 2 of the paper.

A.1 The non-linear external constraint

I consider a standard two-country model with two tradable, but imperfectly substitutable, goods and flexible prices; the model is treated in real terms. A representative household maximizes a time separable utility function defined over consumption. The lifetime utility is

$$\mathbb{E}_t \sum_{i=0}^{\infty} \theta_t^i U\left(C_{t+i}\right)$$

where $\mathbb{E}_t$ is the expectation operator.[23] In the model used for the simulation, the intertemporal discount factor is assumed to follow an Uzawa-type adjustment mechanism $\theta_t = \theta_{t-1}\beta_t$, in which β_t responds the past level of consumption. The online *Supplementary Material* talks about this point more in detail, in this Appendix I focus on the derivation of the budget constraint decomposition only.

The optimality conditions of the representative agent maximization problem equalize the stochastic discount factor to the intertemporal ratio of marginal utilities

$$\Lambda_{t,t+1} = \beta_t \frac{U'\left(C_{t+1}\right)}{U'\left(C_t\right)}$$

and link asset returns to the SDF through the portfolio conditions

$$\mathbb{E}_t\left[\Lambda_{t,t+1} R_{t+1}^i\right] = 1 \qquad i = A, L \tag{A1}$$

Using (A1) in equation (1) to highlight the recursive structure of the budget constraint, we obtain

$$\mathbb{E}_t\left[\Lambda_{t,t+1}\left(R_{t+1}^L L_t - R_{t+1}^A A_t\right)\right] = R_t^L L_{t-1} - R_t^A A_{t-1} - D_t \tag{A2}$$

Recursively substituting for $R_t^L L_{t-1} - R_t^A A_{t-1}$ in (A2) and using again (1) at time t, we derive (2).

[23] The introduction of domestic sub-varieties or price stickiness can be pursued following one of the many examples in the open economy DSGE literature, as in Gali and Monacelli (2008) for instance. It would be fairly simple to extend the model in order to consider a welfare-maximizer, benevolent government by including G_t in the utility function of the household. If the utility is assumed separable in this argument, the implications for the empirical definition of the discount factor would be the same.

Each of the $\mathbb{E}_t\Lambda_{t,t+i}D_{t+i}$ terms on the right hand side of (2) can be further factored into two components

$$\mathbb{E}_t\Lambda_{t,t+i}D_{t+i} = cov_t\left(\Lambda_{t,t+i}, D_{t+i}\right) + \mathbb{E}_t\Lambda_{t,t+i}\mathbb{E}_tD_{t+i} \tag{A3}$$

Using (1) to substitute for the trade balance in the covariance term of (A3), we obtain

$$cov_t\left(\Lambda_{t,t+i}, D_{t+i}\right) = cov_t\left(\Lambda_{t,t+i}, R^L_{t+i}L_{t+i-1}\right) - cov_t\left(\Lambda_{t,t+i}, R^A_{t+i}A_{t+i-1}\right) - cov_t\left(\Lambda_{t,t+i}, B_{t+i}\right)$$

in which the term $cov_t\left(\Lambda_{t,t+i}, R^L_{t+i}L_{t+i-1}\right) - cov_t\left(\Lambda_{t,t+i}, R^A_{t+i}A_{t+i-1}\right)$ captures the effects of returns due to the differences in their risk premia. Each of the terms on the right hand side of this equation can be conveniently expressed as follows. Let us work on $cov_t\left(\Lambda_{t,t+i}, R^L_{t+i}L_{t+i-1}\right)$ first:

$$\begin{aligned} cov_t\left(\Lambda_{t,t+i}, R^L_{t+i}L_{t+i-1}\right) &= \mathbb{E}_t\Lambda_{t,t+i}R^L_{t+i}L_{t+i-1} - \mathbb{E}_t\Lambda_{t,t+i}\mathbb{E}_tR^L_{t+i}L_{t+i-1} \\ &= \mathbb{E}_t\Lambda_{t,t+i-1}L_{t+i-1} - \mathbb{E}_t\Lambda_{t,t+i}\mathbb{E}_tR^L_{t+i}L_{t+i-1} \end{aligned}$$

where the second equality is obtained by using the law of iterated expectations, the definition of the SDF, $\Lambda_{t,t+i} = \Lambda_{t,t+i-1}\Lambda_{t+i-1,t+i}$, and the optimal portfolio condition in $t+i-1$, $\mathbb{E}_{t+i-1}\Lambda_{t+i-1,t+i}R^L_{t+i} = 1$. The same expression holds for $cov_t\left(\Lambda_{t,t+i}, R^A_{t+i}A_{t+i-1}\right)$ too

$$cov_t\left(\Lambda_{t,t+i}, R^A_{t+i}A_{t+i-1}\right) = \mathbb{E}_t\Lambda_{t,t+i-1}A_{t+i-1} - \mathbb{E}_t\Lambda_{t,t+i}\mathbb{E}_tR^A_{t+i}A_{t+i-1}$$

Similarly for $cov_t\left(\Lambda_{t,t+i}, B_{t+i}\right)$ we have

$$cov_t\left(\Lambda_{t,t+i}, B_{t+i}\right) = \mathbb{E}_t\Lambda_{t,t+i}B_{t+i} - \mathbb{E}_t\Lambda_{t,t+i}\mathbb{E}_t\Lambda_{t+i,t+i+1}R^L_{t+i+1}L_{t+i} + \mathbb{E}_t\Lambda_{t,t+i}\mathbb{E}_t\Lambda_{t+i,t+i+1}R^A_{t+i+1}A_{t+i}$$

where the definition of time $t+i$ debt, $B_{t+i} = L_{t+i} - A_{t+i} = \mathbb{E}_{t+i}\Lambda_{t+i,t+i+1}\left(R^L_{t+i+1}L_{t+i} - R^A_{t+i+1}A_{t+i}\right)$, has been applied.

Consider now the sum of the covariances for two consecutive periods $t+i$ and $t+i+1$, with $i > 1$; these terms will appear in the summation in (2) as part of the intertemporal solution of the constraint. After a couple of simplifications, one gets

$$cov_t\left(\Lambda_{t,t+i},D_{t+i}\right)+cov_t\left(\Lambda_{t,t+i+1},D_{t+i+1}\right)=$$

$$=\mathbb{E}_t\Lambda_{t,t+i-1}B_{t+i-1}-\mathbb{E}_t\Lambda_{t,t+i}\mathbb{E}_t\left(R^L_{t+i}L_{t+i-1}-R^A_{t+i}A_{t+i-1}\right) \tag{A4}$$

$$+\mathbb{E}_t\Lambda_{t,t+i}\mathbb{E}_t\Lambda_{t+i,t+i+1}\left(R^L_{t+i+1}L_{t+i}-R^A_{t+i+1}A_{t+i}\right)-\mathbb{E}_t\Lambda_{t,t+i+1}\mathbb{E}_t\left(R^L_{t+i+1}L_{t+i}-R^A_{t+i+1}A_{t+i}\right) \tag{A5}$$

$$+\mathbb{E}_t\Lambda_{t,t+i+1}\mathbb{E}_t\Lambda_{t+i+1,t+i+2}\left(R^L_{t+i+2}L_{t+i+1}-R^A_{t+i+2}A_{t+i+1}\right)-\mathbb{E}_t\Lambda_{t,t+i+1}B_{t+i-1} \tag{A6}$$

We can focus our attention only on the two terms of line (A5). In fact, given the recursive structure of the summation, the first term of line (A4) and last of line (A6) will cancel out with the corresponding terms from periods $t+i-1$ and $t+i+2$. Similarly, the second term of line (A4) and first of line (A6) can be conveniently paired to the matching terms from $t+i-1$ and $t+i+2$ in order to form other lines equivalent to (A5) for the previous and the following pairs of consecutive periods. Line (A5) is then re-written as

$$\frac{1}{R^F_{t,t+i}}\mathbb{E}_t\left(L_{t+i}-A_{t+i}\right)-\frac{1}{R^F_{t,t+i+1}}\mathbb{E}_t\left(R^L_{t+i+1}L_{t+i}-R^A_{t+i+1}A_{t+i}\right) \tag{A7}$$

where the law of iterated expectations and the risk-free representation of the expected SDF have been applied. Notice that the existence of a (multi-period) risk-free rate that can be used to replace the SDF will be the only assumption necessary to obtain the decomposition of the external constraint in (4). Using the fact that $R^F_{t,t+i}=R^F_{t,t+i-1}R^F_{t+i}$ and the definitions of excess and average returns, $R^X_t=R^A_t-R^L_t$ and $R^M_t=\frac{R^A_t+R^L_t}{2}$, and average gross holdings, $B^M_t=\frac{A_t+L_t}{2}$, equation (A7) can be equivalently expressed as

$$\frac{1}{R^F_{t,t+i+1}}\mathbb{E}_t\left[B_{t+i}\left(R^F_{t+i+1}-R^M_{t+i+1}\right)+B^M_{t+i}R^X_{t+i+1}\right] \tag{A8}$$

This result holds for any period pair after $t+1$; period $t+1$ is particular because it is the first period of the summation. However, it is straightforward to show that

$$cov_t\left(\Lambda_{t,t+1},D_{t+1}\right) = L_t-\frac{L_t}{R^F_{t+1}}\mathbb{E}_tR^L_{t+1}-A_t+\frac{A_t}{R^F_{t+1}}\mathbb{E}_tR^A_{t+1} \tag{A9}$$

$$-\frac{1}{R^F_{t+1}}\mathbb{E}_tB_{t+1}+\frac{1}{R^F_{t+2}}\mathbb{E}_t\left(R^L_{t+2}L_{t+1}-R^A_{t+2}A_{t+1}\right) \tag{A10}$$

As seen above, the two terms in line (A10) merge with those in (A4) for period $t+2$ and form the summation term (A8) for the $t+1/t+2$ pair. The two terms in (A9) simply return the first term of the summation

$$\frac{1}{R^F_{t+1}}\mathbb{E}_t\left[B_t\left(R^F_{t+1}-R^M_{t+1}\right)+B^M_tR^X_{t+1}\right] \tag{A11}$$

Finally, using (A8), (A11), and (A3) in (2), we obtain the final expression for the solution of the external constraint in (4).

A.2 First order approximation of the external constraint

In this section, I present more formally the steps necessary to obtain the first order linear approximation of the intertemporal budget constraint and the results in Section 2.3. There are a few useful definitions that it is convenient to recall here: $\hat{B}_t = L_{ss}\hat{L}_t - A_{ss}\hat{A}_t$, $\hat{R}^*_t = A_{ss}\hat{R}^A_t - L_{ss}\hat{R}^L_t = B^M_{ss}\hat{R}^X_t - B_{ss}\hat{R}^M_t$, $\hat{R}^X_t = \hat{R}^A_t - \hat{R}^L_t$ and $\hat{R}^M_t = \frac{\hat{R}^A_t + \hat{R}^L_t}{2}$, $\hat{\mathbb{B}}_t = \hat{B}_{t-1} - \hat{R}^*_t$, $\hat{D}_t = X_{ss}\hat{X}_t - M_{ss}\hat{M}_t$.

Let us start considering a first order approximation of version (A2) of the external budget constraint in period t around the same steady state defined in Section 2.3

$$\mathbb{E}_t\left[L_{ss}\left(\hat{\Lambda}_{t,t+1} + \hat{L}_t + \hat{R}^L_{t+1}\right) - A_{ss}\left(\hat{\Lambda}_{t,t+1} + \hat{A}_t - \hat{R}^A_{t+1}\right)\right] =$$

$$= \beta^{-1}\left(L_{ss}\hat{L}_{t-1} + L_{ss}\hat{R}^L_t - A_{ss}\hat{A}_{t-1} - A_{ss}\hat{R}^A_t\right) - \left(X_{ss}\hat{X}_t - M_{ss}\hat{M}_t\right)$$

in a more compact form

$$\mathbb{E}_t\left[B_{ss}\hat{\Lambda}_{t,t+1} + \hat{\mathbb{B}}_{t+1}\right] = \beta^{-1}\hat{\mathbb{B}}_t - \hat{D}_t$$

which can be expressed for $\hat{\mathbb{B}}_t$ as

$$\hat{\mathbb{B}}_t = \beta\mathbb{E}_t\left[\hat{\Lambda}_{t,t+1} + \hat{\mathbb{B}}_{t+1}\right] + \beta\hat{D}_t \tag{A12}$$

Solving (A12) by forward recursive substitution and substitute the $t+1$ constraint into period t constraint, after applying the transversality condition $\lim_{T\to\infty}\beta^T\mathbb{E}_t\hat{\mathbb{B}}_{t+T}$ which is trivially satisfied by stationary log-deviations, we get (6)

$$\hat{B}_t = \mathbb{E}_t\sum_{i=1}^{\infty}\beta^{i-1}\left(B_{ss}\hat{\Lambda}_{t+i-1,t+i} + \beta\hat{D}_{t+i}\right)$$

where use is made of the fact that in period t $\hat{\mathbb{B}}_t = \beta\left(\hat{B}_t - \hat{D}_t\right)$ as well.

Similar steps can be followed to derive the linearized constraint (5) from version (1) of the external constraint. The linearization in period t gives

$$\hat{B}_{t-1} = \beta\left(\hat{B}_t + \hat{D}_t\right) + \hat{R}^*_t$$

lead this equation one period ahead and simply apply the expectations

$$\hat{B}_t = \beta\mathbb{E}_t\left(\hat{B}_{t+1} + \hat{D}_{t+1}\right) + \mathbb{E}_t\hat{R}^*_{t+1}$$

substituting for $\hat{B}_t$ in the time t constraint we get

$$\hat{B}_{t-1} = \beta\hat{D}_t + \beta^2\mathbb{E}_t\hat{D}_{t+1} + \hat{R}^*_t + \beta\mathbb{E}_t\hat{R}^*_{t+1} + +\beta^2\mathbb{E}_t\hat{B}_{t+1}$$

Recursive iteration over $\hat{B}$, after imposing the transversality condition $\lim_{T\to\infty}\beta^T\mathbb{E}_t\hat{B}_{t+T}$ and using the time t constraint again to substitute for $\hat{B}_{t-1}$, leads to the final intertemporal budget constraint corresponding to (5)

$$\hat{B}_t = \mathbb{E}_t\sum_{i=1}^{\infty}\beta^{i-1}\left(\hat{R}^*_{t+i} + \beta\hat{D}_{t+i}\right)$$

Finally, the first order linearization of the portfolio conditions (A1) implies $\mathbb{E}_t\hat{R}^X_{t+1} = 0$. Using this condition and the definition of $\hat{R}^*_t$, after equalizing (6) to (5), we obtain the result in (7).